Variational Methods: The ABS(X) Potential in Momentum Space

The energy operator in atomic units in coordinate space for a unit mass particle with potential energy V = |x| is given below.

$$H = -\frac{1}{2}\cdot\frac{d^2}{dx^2}\blacksquare + |x|\cdot\blacksquare$$

Suggested trial wave function: $\Psi(x,\beta) := \left(\frac{2\cdot\beta}{\pi}\right)^{\frac{1}{4}}\cdot\exp(-\beta\cdot x^2)$

Demonstrate that the wave function is normalized. $\int_{-\infty}^{\infty}\Psi(x,\beta)^2\,dx$ assume, $\beta > 0 \rightarrow 1$

Carry out Fourier transform to get momentum wave function:

$$\Phi(p,\beta) := \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{-\infty}^{\infty}\exp(-i\cdot p\cdot x)\cdot\Psi(x,\beta)\,dx \;\Big|\; \begin{array}{l}\text{assume}, \beta > 1\\ \text{simplify}\end{array} \rightarrow \frac{1}{2}\cdot\frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{4}}}\cdot\frac{e^{\frac{-1}{4}\cdot\frac{p^2}{\beta}}}{\beta^{\frac{1}{4}}}$$

Demonstrate that the momentum wave function is normalized. $\int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\Phi(p,\beta)\,dp$ assume, $\beta > 0 \rightarrow 1$

The energy operator in momentum space is: $H = \frac{p^2}{2}\cdot\blacksquare + \left|i\cdot\frac{d}{dp}\blacksquare\right|$

Evaluate the variational energy integral.

$$E(\beta) := \int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\frac{p^2}{2}\cdot\Phi(p,\beta)\,dp \ldots + \int_{-\infty}^{\infty}\overline{\Phi(p,\beta)}\cdot\left|i\cdot\frac{d}{dp}\Phi(p,\beta)\right|\,dp \;\Big|\; \begin{array}{l}\text{assume}, \beta > 0\\ \text{simplify}\end{array} \rightarrow \frac{1}{2}\cdot\frac{\pi^{\frac{1}{2}}\cdot\beta^{\frac{3}{2}} + 2^{\frac{1}{2}}}{\beta^{\frac{1}{2}}\cdot\pi^{\frac{1}{2}}}$$

Minimize the energy with respect to the variational parameter β and report its optimum value and the ground-state energy.

$\beta := 1$ $\quad\beta := \text{Minimize}(E,\beta)$ $\quad\beta = 0.542$ $\quad E(\beta) = 0.813$

Plot the coordinate and momentum wave functions and the potential energy on the same graph.