Approximate Methods: Particle in a Gravitational Field

The following problem deals with a particle of mass, μ, in a gravitational field with acceleration due to gravity equal to g.

Energy operator for particles near Earth's surface: $-\frac{1}{2\cdot\mu}\cdot\frac{d^2}{dz^2}\blacksquare + m\cdot g\cdot z\cdot\blacksquare$

Trial wave function: $\Psi(\alpha,z) := 2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)$

Demonstrate that the wave function is normalized:

$$\int_0^{\infty}\left(2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)\right)^2 dz \text{ assume}, \alpha > 0 \;\rightarrow 1$$

Evaluate variational integral:

$$E(\alpha,g,\mu) := \int_0^{\infty} 2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)\cdot\frac{-1}{2\cdot\mu}\cdot\frac{d^2}{dz^2}\left(2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)\right) dz \;\ldots\; \text{assume}, \alpha > 0 \;\rightarrow \frac{1}{2}\cdot\frac{\alpha^2}{\mu} + \frac{3}{2}\cdot\mu\cdot\frac{g}{\alpha}$$
$$+ \int_0^{\infty}\left(2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)\right)^2\cdot\mu\cdot g\cdot z\, dz$$

Minimize energy with respect to variational parameter α:

$$\frac{d}{d\alpha}\left(\frac{1}{2}\cdot\frac{\alpha^2}{\mu} + \frac{3}{2\cdot\alpha}\cdot\mu\cdot g\right) = 0 \text{ solve}, \alpha \;\rightarrow \begin{bmatrix} \frac{1}{2}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} \\ \frac{-1}{4}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} + \frac{1}{4}\cdot i\cdot 3^{\frac{1}{2}}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} \\ \frac{-1}{4}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} - \frac{1}{4}\cdot i\cdot 3^{\frac{1}{2}}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} \end{bmatrix}$$

Substitute optimum α (real root) into expression for energy:

$$E(g,\mu) := \frac{1}{2}\cdot\frac{\alpha^2}{\mu} + \frac{3}{2\cdot\alpha}\cdot\mu\cdot g \;\left|\begin{array}{l}\text{substitute}, \alpha = \frac{1}{2}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}} \\ \text{simplify}\end{array}\right. \rightarrow \frac{3}{8}\cdot\mu\cdot g\cdot\frac{12^{\frac{2}{3}}}{\left(\mu^2\cdot g\right)^{\frac{1}{3}}}$$

Evaluate the energy for μ = 1 and g = 1. Compare your result to the exact energy, 1.856.

$\mu := 1$ $g := 1$ $E(g,\mu) = 1.966$ $\frac{E(g,\mu) - 1.856}{1.856}\cdot 100 = 5.903$

Plot the wave function: $\alpha := \frac{1}{2}\cdot 12^{\frac{1}{3}}\cdot\left(\mu^2\cdot g\right)^{\frac{1}{3}}$ $z := 0, .01 .. 10$