Approximate Methods: The Quartic Oscillator in Momentum Space
Frank Rioux

For unit mass the quartic oscillator has the following energy operator in atomic units in coordinate space.

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + x^4 \]

Suggested trial wave function:

\[ \Psi(x, \beta) := \left( \frac{2\beta}{\pi} \right)^{\frac{1}{4}} \exp\left(-\beta x^2\right) \]

Demonstrate that the wave function is normalized.

\[ \int_{-\infty}^{\infty} \Psi(x, \beta)^2 dx \text{ assume, } \beta > 0 \rightarrow 1 \]

Fourier transform the coordinate wave function into the momentum representation.

\[ \Phi(p, \beta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i p \cdot x) \Psi(x, \beta) dx \]

\[ \text{assume, } \beta > 1 \text{ simplify } \rightarrow \frac{3}{2} - \frac{1}{4} \frac{p^2}{\beta^2} \]

Demonstrate that the momentum wave function is normalized.

\[ \int_{-\infty}^{\infty} \Phi(p, \beta) \cdot \Phi(p, \beta) dp \text{ assume, } \beta > 0 \rightarrow 1 \]

The quartic oscillator energy operator in momentum space:

\[ H = \frac{p^2}{2} \]

Evaluate the variational energy integral.

\[ E(\beta) := \int_{-\infty}^{\infty} \frac{\Phi(p, \beta) \cdot p^2}{2} \cdot \Phi(p, \beta) dp + \int_{-\infty}^{\infty} \frac{\Phi(p, \beta) \cdot d^4}{dp^4} \Phi(p, \beta) dp \]

\[ \text{assume, } \beta > 0 \text{ simplify } \rightarrow \frac{1}{16} \frac{8 \beta^3 + 3}{\beta^2} \]

Minimize the energy with respect to the variational parameter \( \beta \) and report its optimum value and the ground-state energy.

\[ \beta := 1 \quad \beta := \text{Minimize}(E, \beta) = 0.90856 \quad E(\beta) = 0.68142 \]

Plot the coordinate and momentum wave functions and the potential energy on the same graph.
These results demonstrate the uncertainty principle. For the harmonic potential, $x^2/2$, the coordinate and momentum wave functions are identical. Compared to the harmonic potential the quartic potential, $x^4$, constrains the spatial wave function leading to less uncertainty in position. The uncertainty principle, therefore, requires an increase in the momentum uncertainty. This is clearly revealed in graph above.