

## Approximate Methods: The Quartic Oscillator in Momentum Space

For unit mass the quartic oscillator has the following energy operator in atomic units in coordinate space.

$$H = -\frac{1}{2} \cdot \frac{d^2}{dx^2} + x^4 \quad \text{Suggested trial wave function: } \Psi(x, \beta) := \left(\frac{2 \cdot \beta}{\pi}\right)^{\frac{1}{4}} \cdot \exp(-\beta \cdot x^2)$$

Demonstrate that the wave function is normalized.  $\int_{-\infty}^{\infty} \Psi(x, \beta)^2 dx$  assume,  $\beta > 0 \rightarrow 1$

Fourier transform the coordinate wave function into the momentum representation.

$$\Phi(p, \beta) := \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int_{-\infty}^{\infty} \exp(-i \cdot p \cdot x) \cdot \Psi(x, \beta) dx \quad \left| \begin{array}{l} \text{assume, } \beta > 0 \\ \text{simplify} \end{array} \right. \rightarrow \frac{1}{2} \cdot \frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{4}}} \cdot \frac{e^{-\frac{1}{4} \cdot \frac{p^2}{\beta}}}{\beta^{\frac{1}{4}}}$$

Demonstrate that the momentum wave function is normalized.  $\int_{-\infty}^{\infty} \overline{\Phi(p, \beta)} \cdot \Phi(p, \beta) dp$  assume,  $\beta > 0 \rightarrow 1$

The quartic oscillator energy operator in momentum space:  $H = \frac{p^2}{2} + \frac{d^4}{dp^4}$

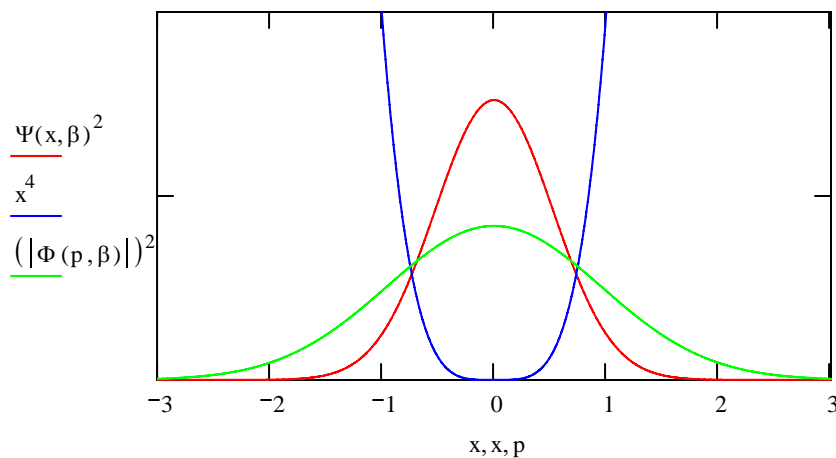
Evaluate the variational energy integral.

$$E(\beta) := \int_{-\infty}^{\infty} \overline{\Phi(p, \beta)} \cdot \frac{p^2}{2} \cdot \Phi(p, \beta) dp + \int_{-\infty}^{\infty} \overline{\Phi(p, \beta)} \cdot \frac{d^4}{dp^4} \Phi(p, \beta) dp \quad \left| \begin{array}{l} \text{assume, } \beta > 0 \\ \text{simplify} \end{array} \right. \rightarrow \frac{1}{16} \cdot \frac{8 \cdot \beta^3 + 3}{\beta^2}$$

Minimize the energy with respect to the variational parameter  $\beta$  and report its optimum value and the ground-state energy.

$$\beta := 1 \quad \beta := \text{Minimize}(E, \beta) \quad \beta = 0.90856 \quad E(\beta) = 0.68142$$

Plot the coordinate and momentum wave functions and the potential energy on the same graph.



These results demonstrate the uncertainty principle. For the harmonic potential,  $x^2/2$ , the coordinate and momentum wave functions are identical. Compared to the harmonic potential the quartic potential,  $x^4$ , constrains the spatial wave function leading to less uncertainty in position. The uncertainty principle, therefore, requires an increase in the momentum uncertainty. This is clearly revealed in graph above.