Fourier Synthesis

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The purpose of this tutorial is to use Dirac notation to examine Fourier synthesis. The first step is to write the function symbolically in Dirac notation.

\[ f(x) = \langle x \mid f \rangle \] (1)

Select an orthonormal basis set, |n>, for which the completeness relation holds.

\[ \sum_n |n\rangle\langle n| = 1 \] (2)

Expand |f> in terms of |n> by inserting equation (2) into the right side of equation (1). In other words write f(x) as a weighted \(\langle x \mid n \rangle \) superposition using the \(\langle x \mid n \rangle \) basis set (the |n> basis set expressed in the coordinate representation).

\[ f(x) = \sum_n \langle x \mid n \rangle \langle n \mid f \rangle \] (3)

Evaluate the Fourier coefficient, \(\langle n \mid f \rangle \), using the continuous completeness relation in coordinate space.

\[ \int |x'\rangle \langle x'| dx' = 1 \] (4)

Equation (3) becomes,

\[ f(x) = \sum_n \langle x \mid n \rangle \int |n\rangle\langle x'| \langle x'| f \rangle dx' \] (5)

Now select a function

\[ \langle x' \mid f \rangle = x'^3(1 - x') \] (6)

over the interval (0,1). Choose the following orthonormal basis set over the same interval.

\[ \langle x \mid n \rangle = \sqrt{2} \sin(n\pi x) \] (7)

Substitution of equations (6) and (7) into (5) yields

\[ f(x) = \sum_n \sqrt{2} \sin(n\pi x) \int_0^1 \sqrt{2} \sin(n\pi x')x'^3(1 - x') dx' \] (8)
The Fourier synthesis and the original function are shown for \( n = 2, 4, \) and 10 in the figure below.

\[
x := 0, .025 \ldots 1.0 \quad f(x, n) := \sum_{i=1}^{n} \left[ \sqrt{2} \cdot \sin(i \cdot \pi \cdot x) \cdot \int_{0}^{1} \sqrt{2} \cdot \sin(i \cdot \pi \cdot x') \cdot x'^{3} \cdot (1 - x') \, dx' \right]
\]