The Dirac Delta Function

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The Dirac delta function expressed in Dirac notation is: \( \Delta(x - x_1) = \langle x \mid x_1 \rangle \). The \( \langle x \mid x_1 \rangle \) bracket is evaluated using the momentum completeness condition. See the Mathematical Appendix for definitions of the required Dirac brackets and other mathematical tools used in the analysis that follows.

\[
\langle x \mid x_1 \rangle = \int_{-\infty}^{\infty} \langle x \mid p \rangle \langle p \mid x_1 \rangle dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ipx)\exp(-ipx_1)dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[ip(x - x_1)]dp
\]

Evaluation of this integral over a finite range of momentum values shows that the delta function is small except in the immediate neighborhood of \( x_1 \). Integrating from -20 to 20 to reduce computational time shows that \( \langle x \mid x_1=2 \rangle \) is small except in the area \( x = 2 \).

\[
x_1 := 2 \quad x := 0, 0.01 \ldots 4 \quad \text{Dirac}(x, x_1) := \frac{1}{2\pi} \int_{-20}^{20} \exp[i\cdot p\cdot(x - x_1)] \, dp
\]

The Fourier transform of the Dirac delta function into the momentum representation yields the following result.

\[
\int_{-\infty}^{\infty} \langle p \mid x \rangle \langle x \mid x_1 \rangle \, dx = \frac{1}{\sqrt{2\pi}} \exp(-ipx_1) = \langle p \mid x_1 \rangle
\]

The normalization constant is omitted for clarity of expression and the previous value of \( x_1 \) is cleared to allow symbolic calculation.

\[
x_1 := x_1 \quad \int_{-\infty}^{\infty} \exp(-i\cdot p\cdot x) \cdot \Delta(x - x_1) \, dx \, \text{simplify} \rightarrow e^{-p\cdot x_1\cdot i}
\]
Mathematical Appendix

The position and momentum completeness conditions:

\[ \int |x\rangle \langle x| dx = 1 \quad \int |p\rangle \langle p| dp = 1 \]

The momentum eigenstate in the coordinate representation:

\[ \langle x | p \rangle = \frac{1}{\sqrt{2\pi}} \exp(ipx) \]

The position eigenstate in the momentum representation:

\[ \langle p | x \rangle = \frac{1}{\sqrt{2\pi}} \exp(-ipx) \]