

Fourier Synthesis

Frank Rioux
Department of Chemistry
College of St. Benedict | St. John's University

The purpose of this tutorial is to use Dirac notation to examine Fourier synthesis. The first step is to write the function symbolically in Dirac notation.

$$f(x) = \langle x | f \rangle \quad (1)$$

Select an orthonormal basis set, $|n\rangle$, such that the finite completeness relation holds.

$$\sum_n |n\rangle \langle n| = 1 \quad (2)$$

Expand $|f\rangle$ in terms of $|n\rangle$ by inserting equation (2) into the right side of equation (1).

$$f(x) = \sum_n \langle x | n \rangle \langle n | f \rangle \quad (3)$$

Evaluate the Fourier coefficient, $\langle n | f \rangle$, using the continuous completeness relation in coordinate space.

$$\int |x'\rangle \langle x'| dx' = 1 \quad (4)$$

Equation (3) becomes,

$$f(x) = \sum_n \langle x | n \rangle \int \langle n | x' \rangle \langle x' | f \rangle dx' \quad (5)$$

Now assume

$$\langle x | f \rangle = x^3 (1 - x) \quad (6)$$

over the interval (0,1). Choose the following orthonormal basis set over the same interval.

$$\langle x | n \rangle = \sqrt{2} \sin(n\pi x) \quad (7)$$

Substitution of equations (6) and (7) into (5) yields

$$f(x) = \sum_n \sqrt{2} \sin(n\pi x) \int_0^1 \sqrt{2} \sin(n\pi x') x'^3 (1 - x') dx' \quad (8)$$

The Fourier synthesis and the original function are shown for $n = 2, 4,$ and 10 in the figure below.

$$x := 0, .025 .. 1.025 \quad f(x, n) := \sum_{i=1}^n \left[\sqrt{2} \cdot \sin(i \cdot \pi \cdot x) \cdot \int_0^1 \sqrt{2} \cdot \sin(i \cdot \pi \cdot xp) \cdot xp^3 \cdot (1 - xp) \, dxp \right]$$

