Analyzing Two-Photon Interferometry Using Mathcad and Tensor Algebra

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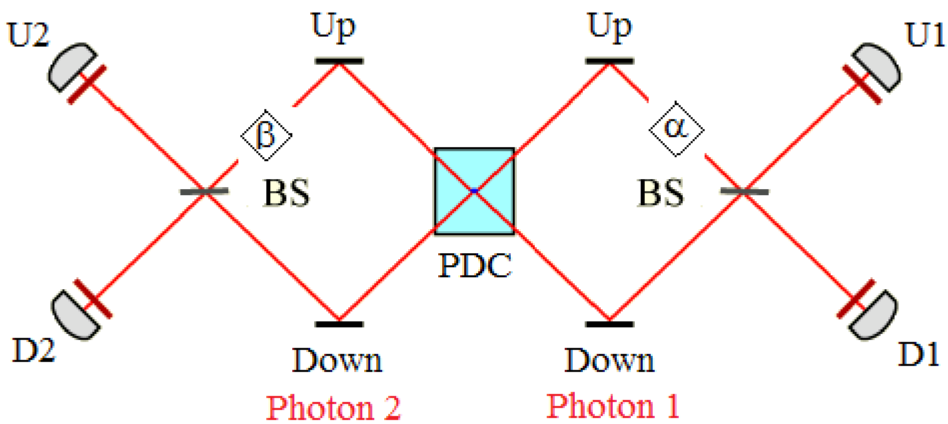
Greenberger, Horne and Zeilinger (GHZ) surveyed the then relatively new field of multiparticle interferometry in their August 1993 *Physics Today* article, "Multiparticle Interferometry and the Superposition Principle." This tutorial will use Mathcad and tensor algebra to analyze the results associated with Figure 2, shown below.



Richard Feynman identified the superposition principle, as manifested in the double-slit experiment, as the *heart of quantum mechanics and its* only *mystery*. Indeed this simple example of a single-photon superposition illustrates a number of quantum fundamentals, but as GHZ point out multi-photon entangled superpositions reveal additional mysteries such as quantum correlations that challenge philosophical positions such as local realism.

The operators required to analyze this two-photon interferometer are defined below. α and β are phase shifters.

Identity:	Mirror:	Phase shift:	Beam splitter:
$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$A(\delta) := \begin{pmatrix} e^{i \cdot \delta} & 0 \\ 0 & 1 \end{pmatrix}$	$BS := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

The parametric down converter (PDC) creates two photons traveling in opposite directions in the arms of the interferometer, with photon 1 moving to the right and photon 2 to the left. The initial state is the following entangled superposition.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|u\rangle_1 |d\rangle_2 + |d\rangle_1 |u\rangle_2\right] \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

There are four output states. The appendix shows how the input and output states are constructed in Mathcad.

$$|uu\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |ud\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |du\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |dd\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The input and output states are defined in Mathcad syntax.

$$u := \begin{pmatrix}1\\0\end{pmatrix} \qquad d := \begin{pmatrix}0\\1\end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \qquad u1u2 := \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \; u1d2 := \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \; d1u2 := \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \; d1d2 := \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The four outcome probabilities are now calculated. The appendix shows how Mathcad is used to carry out tensor products of matrices.

$$P_{u1u2}(\alpha,\beta) := \left(\left|u1u2^T\cdot \text{kronecker}(BS,BS)\cdot \text{kronecker}(A(\alpha),A(\beta))\cdot \text{kronecker}(M,M)\cdot\Psi\right|\right)^2$$

$$P_{u1d2}(\alpha,\beta) := \left(\left|u1d2^T\cdot \text{kronecker}(BS,BS)\cdot \text{kronecker}(A(\alpha),A(\beta))\cdot \text{kronecker}(M,M)\cdot\Psi\right|\right)^2$$

$$P_{d1u2}(\alpha,\beta) := \left(\left|d1u2^T\cdot \text{kronecker}(BS,BS)\cdot \text{kronecker}(A(\alpha),A(\beta))\cdot \text{kronecker}(M,M)\cdot\Psi\right|\right)^2$$

$$P_{d1d2}(\alpha,\beta) := \left(\left|d1d2^T\cdot \text{kronecker}(BS,BS)\cdot \text{kronecker}(A(\alpha),A(\beta))\cdot \text{kronecker}(M,M)\cdot\Psi\right|\right)^2$$

The probability that the individual detectors will fire is given by the following sums.

$$P_{u1} = P_{u1u2} + P_{u1d2} \qquad P_{d1} = P_{d1d2} + P_{d1u2} \qquad P_{u2} = P_{u1u2} + P_{d1u2} \qquad P_{d2} = P_{d1d2} + P_{u1d2}$$

When there is no phase difference in the two branches of the interferometer there is a strong correlation in the detector responses. Fifty percent of the time the photons arrive at the up-detectors and 50% of the time at the down-detectors. There are no coincidences between an up- and a down-detector. This might be called bosonic behavior - bosons like to do the same thing.

$$\alpha := 0\cdot deg \qquad \beta := 0\cdot deg \qquad \begin{pmatrix}P_{u1u2}(\alpha,\beta) & P_{u1d2}(\alpha,\beta)\\ P_{d1u2}(\alpha,\beta) & P_{d1d2}(\alpha,\beta)\end{pmatrix} = \begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}$$

A 90 degree phase difference results in no correlation. The four detector pairs fire with equal frequency.

$$\alpha := 0\cdot deg \qquad \beta := 90\cdot deg \qquad \begin{pmatrix}P_{u1u2}(\alpha,\beta) & P_{u1d2}(\alpha,\beta)\\ P_{d1u2}(\alpha,\beta) & P_{d1d2}(\alpha,\beta)\end{pmatrix} = \begin{pmatrix}0.25 & 0.25\\ 0.25 & 0.25\end{pmatrix}$$

A 180 degree phase difference results in what might be called fermionic behavior. If one photon is detected at an up-detector, the other is registered at a down detector. They never both arrive at the same type of detector. Fermions don't like to do the same thing at the same time.

$$\alpha := 0 \cdot \mathrm{deg} \qquad \beta := 180 \cdot \mathrm{deg} \qquad \begin{pmatrix} P_{u1u2}(\alpha,\beta) & P_{u1d2}(\alpha,\beta) \\ P_{d1u2}(\alpha,\beta) & P_{d1d2}(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}$$

Intermediate correlation is achieved with a 30 degree phase difference.

$$\alpha := 0 \cdot \mathrm{deg} \qquad \beta := 30 \cdot \mathrm{deg} \qquad \begin{pmatrix} P_{u1u2}(\alpha,\beta) & P_{u1d2}(\alpha,\beta) \\ P_{d1u2}(\alpha,\beta) & P_{d1d2}(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} 0.467 & 0.033 \\ 0.033 & 0.467 \end{pmatrix}$$

Note that the sums of the rows and the sums of the columns always equal 1/2.

$$P_{1u} = P_{2u} = P_{1d} = P_{2d} = \frac{1}{2}$$

An individual detector fires 50% of the time in spite of the correlations that may occur between two detectors. Looking at individual detectors reveals totally random behavior. It is only when coincidences between pairs of detectors on the left and right are examined, are correlations observed.

Appendix

The tensor product of two vectors is shown below.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Mathcad does not have a command for the vector tensor product, so it is necessary to develop a way of implementing it using *kronecker*, which requires square matrices. For this reason the spin vector is stored in the left column of a 2x2 matrix by augmenting the spin vector with the null vector. After the matrix tensor products have been carried out using *kronecker* the final spin vector resides in the left column of the final square matrix. Next the *submatrix* command is used to save this column, discarding the rest of the matrix.

The Mathcad syntax for the tensor multiplication of two vectors is as follows.

$$\Psi(a,b) := \mathrm{submatrix}\left[\mathrm{kronecker}\left[\mathrm{augment}\left[a, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right], \mathrm{augment}\left[b, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]\right], 1, 4, 1, 1\right]$$

The initial photon state: $\frac{1}{\sqrt{2}} \cdot (\Psi(u,d) + \Psi(d,u)) = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix}$

Output states: $\Psi(u,u) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\Psi(u,d) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\Psi(d,u) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\Psi(d,d) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Tensor matrix multiplication is also known as Kronecker multiplication. Here it is shown that the Mathcad result using the *kronecker* command is identical to the hand calculation.

$$\mathrm{kronecker}(BS, BS) = \begin{pmatrix} 0.5 & 0.5i & 0.5i & -0.5 \\ 0.5i & 0.5 & -0.5 & 0.5i \\ 0.5i & -0.5 & 0.5 & 0.5i \\ -0.5 & 0.5i & 0.5i & 0.5 \end{pmatrix}$$

$$\widehat{BS} \otimes \widehat{BS} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} & i\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ i\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i & i & -1 \\ i & 1 & -1 & i \\ i & -1 & 1 & i \\ -1 & i & i & 1 \end{pmatrix}$$