

Entanglement Reveals a Conflict Between Local Realism and Quantum Theory

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A tensor algebra approach is used to demonstrate the challenge to the local realistic position of reality that quantum mechanical entanglement creates. The example is drawn from Chapter 3 of David Z Albert's text, *Quantum Mechanics and Experience*.

A quon (any entity that exhibits both wave and particle aspects in the peculiar quantum manner - Nick Herbert, *Quantum Reality*, page 64) has a variety of properties each of which can take on two values. For example, it has the property of *hardness* and can be either *hard* or *soft*. It also has the property of *color* and can be either *black* or *white*.

In the matrix formulation of quantum mechanics these states are represented by the following vectors.

$$\text{Hard} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Soft} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{Black} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{White} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Hard and *Soft* represent an orthonormal basis in the two-dimensional *Hardness* vector space.

$$\text{Hard}^T \cdot \text{Hard} = 1 \quad \text{Soft}^T \cdot \text{Soft} = 1 \quad \text{Hard}^T \cdot \text{Soft} = 0$$

Likewise *Black* and *White* are an orthonormal basis in the two-dimensional *Color* vector space.

$$\text{Black}^T \cdot \text{Black} = 1 \quad \text{White}^T \cdot \text{White} = 1 \quad \text{Black}^T \cdot \text{White} = 0$$

The relationship between the two bases is reflected in the following projection calculations.

$$\text{Hard}^T \cdot \text{Black} \rightarrow \frac{\sqrt{2}}{2} \quad \text{Hard}^T \cdot \text{White} \rightarrow \frac{\sqrt{2}}{2} \quad \text{Soft}^T \cdot \text{Black} \rightarrow \frac{\sqrt{2}}{2} \quad \text{Soft}^T \cdot \text{White} \rightarrow -\frac{\sqrt{2}}{2}$$

Clearly *Black* and *White* can be written as superpositions of *Hard* and *Soft*, and vice versa.

$$\begin{aligned} \frac{1}{\sqrt{2}} \cdot (\text{Hard} + \text{Soft}) &\rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} & \frac{1}{\sqrt{2}} \cdot (\text{Hard} - \text{Soft}) &\rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \cdot (\text{Black} + \text{White}) &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \frac{1}{\sqrt{2}} \cdot (\text{Black} - \text{White}) &\rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Hard, *Soft*, *Black* and *White* are measurable properties and the vectors representing them are eigenstates of the *Hardness* and *Color* operators with eigenvalues +/- 1.

Operators

$$\text{Hardness} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Color} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalue +1

$$\text{Hardness} \cdot \text{Hard} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{Color} \cdot \text{Black} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Eigenvalue -1

$$\text{Hardness} \cdot \text{Soft} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\text{Color} \cdot \text{White} \rightarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Hard and *Soft* are not eigenfunctions of the *Color* operator, and *Black* and *White* are not eigenfunctions of the *Hardness* operator.

$$\text{Hardness} \cdot \text{Black} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\text{Hardness} \cdot \text{White} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\text{Color} \cdot \text{Hard} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Color} \cdot \text{Soft} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

As the *Hardness-Color* commutator shows, *Hardness* and *Color* are incompatible observables.

$$\text{Hardness} \cdot \text{Color} - \text{Color} \cdot \text{Hardness} \rightarrow \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

We now proceed with an analysis of the implications of the following two-quon entangled state, expressed in tensor format. (The Appendix shows how to set this state up using Mathcad.)

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|\text{Hard}\rangle_1 |\text{Soft}\rangle_2 - |\text{Soft}\rangle_1 |\text{Hard}\rangle_2] = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Given $|\Psi\rangle$ the expectation value for measuring *Hardness* on the first quon is 0. The same is true for the second quon. In other words, it is equally likely for either quon to be *Hard* or *Soft*. (*Kronecker* is Mathcad's command for tensor multiplication of matrices. See the Appendix for more detail.)

$$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{I}) \cdot \Psi = 0$$

$$\Psi^T \cdot \text{kronecker}(\text{I}, \text{Hardness}) \cdot \Psi = 0$$

However, if one quon is found to be *Hard* by measurement, the second will be measured *Soft*, and vice versa. In other words, there is perfect anti-correlation between the joint measurement of this property on the two quons.

$$\Psi^T \cdot \text{kroncker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$$

Given $|\Psi\rangle$ the expectation value for measuring *Color* on the first quon is 0. The same is true for the second quon. In other words, it is equally likely for either quon to be *Black* or *White*.

$$\Psi^T \cdot \text{kroncker}(\text{Color}, \text{I}) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kroncker}(\text{I}, \text{Color}) \cdot \Psi = 0$$

However, if one quon is found to be *Black* by measurement, the second will be measured *White* and vice versa. In other words, there is perfect anti-correlation between the joint measurement of this property on the two quons.

$$\Psi^T \cdot \text{kroncker}(\text{Color}, \text{Color}) \cdot \Psi = -1$$

Furthermore, as the following calculations show, there is no correlation between the measurement outcomes on *Color* and *Hardness*.

$$\Psi^T \cdot \text{kroncker}(\text{Hardness}, \text{Color}) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kroncker}(\text{Color}, \text{Hardness}) \cdot \Psi = 0$$

As the foundation for their belief in local realism, Einstein, Podolsky and Rosen (*EPR*) defined the concept of *element of reality* in their famous 1935 *Physical Review* paper,

"If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an *element of reality* corresponding to this physical quantity."

It would seem from the above results, namely these,

$$\Psi^T \cdot \text{kroncker}(\text{Color}, \text{Color}) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kroncker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$$

that according to *EPR* both hardness and color are *elements of reality*. If the hardness of quon 1 is measured and found to be soft, we know without measurement (given the reliability of quantum mechanical predictions) that quon 2 is hard. Likewise, if the color of quon 2 is measured and found to be white, we know without measurement that quon 1 black

The problem with this interpretation is that it has previously been shown that the *Hardness* and *Color* operators do not commute, meaning that they represent incompatible observables. Incompatible observables cannot be known (determined) simultaneously. A contradiction between the *EPR* reality criterion and quantum mechanics has thus been shown to exist.

Given the reliability of quantum mechanical predictions regarding the outcome of experimental measurements, it appears necessary to abandon the concept of local realism.

Appendix

Tensor multiplication is used to construct the initial state using Mathcad commands *submatrix*, *kroncker* and *augment*.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \text{submatrix} \left[\text{kroncker} \left[\text{augment} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{augment} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], 1, 4, 1, 1 \right] \dots \\ + \text{submatrix} \left[\text{kroncker} \left[\text{augment} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{augment} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], 1, 4, 1, 1 \right] \right] \end{bmatrix} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

Kronecker is the Mathcad command that carries out the tensor multiplication of matrices. For example, consider the tensor multiplication of the *Hardness* and *Color* matrix operators.

$$\text{Hardness} \otimes \text{Color} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\text{kroncker}(\text{Hardness}, \text{Color}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{kroncker} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$