Another Assault on Local Realism

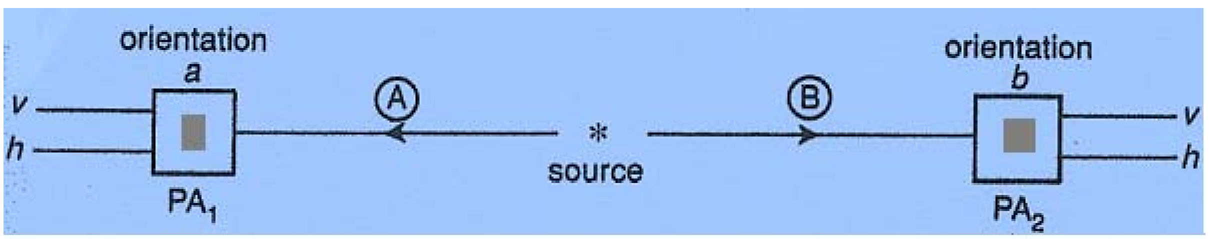
Frank Rioux
Professor Emeritus of Chemistry
CSB|SJU

The purpose of this tutorial is to review Nick Herbert's "simple proof of Bell's theorem" as presented in Chapter 12 of *Quantum Reality* using matrix and tensor algebra.

A two-stage atomic cascade emits entangled photons (A and B) in opposite directions with the same circular polarization according to the observers in their path.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A |L\rangle_B + |R\rangle_A |R\rangle_B\right]$$

The experiment involves the measurement of photon polarization states in the vertical/horizontal measurement basis, and allows that the polarization analyzers (PAs) can be oriented at different angles ***a*** and ***b***.



To dramatize the quantum weirdness of this EPR experiment, Herbert places PA_1 on earth and PA_2 on Betelgeuse, 540 light years away. The source is a space ship midway between the PAs.

In the vertical/horizontal measurement basis the initial polarization state is (see the Appendix for a justification),

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_A |V\rangle_B - |H\rangle_A |H\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_A \otimes \begin{pmatrix}1\\0\end{pmatrix}_B - \begin{pmatrix}0\\1\end{pmatrix}_A \otimes \begin{pmatrix}0\\1\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

There are four measurement outcomes: both photons are vertically polarized, both are horizontally polarized, one is vertical and the other horizontal, and vice versa. The tensor representation of these measurement states are provided below.

$$|VV\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |VH\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |HV\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |HH\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We now write all states, Ψ and the measurement states, in Mathcad's vector format.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad VV := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad VH := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad HV := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad HH := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Next, the operator representing the rotation of PA_1 by angle ***a*** clockwise and PA_2 by angle ***b*** counter-clockwise (so that the PAs turn in the same direction) is constructed using matrix tensor multiplication. **Kronecker** is Mathcad's command for tensor matrix multiplication.

$$\text{RotOp}(a,b) := \text{kronecker}\left[\begin{pmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{pmatrix}, \begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix}\right]$$

The probability that the detectors will behave the same or differently is calculated as follows.

$$P_{same}(a,b) := \left(VV^T \cdot \text{RotOp}(a,b) \cdot \Psi\right)^2 + \left(HH^T \cdot \text{RotOp}(a,b) \cdot \Psi\right)^2$$

$$P_{diff}(a,b) := \left(VH^T \cdot \text{RotOp}(a,b) \cdot \Psi\right)^2 + \left(HV^T \cdot \text{RotOp}(a,b) \cdot \Psi\right)^2$$

Now we can get on with some actual calculations. The following calculations show that if the PAs are oriented at the same angle they behave the same way 100% of the time. This is called perfect correlation.

$P_{same}(0 \cdot deg, 0 \cdot deg) = 100\%$ $\qquad P_{same}(30 \cdot deg, 30 \cdot deg) = 100\%$ $\qquad P_{same}(90 \cdot deg, 90 \cdot deg) = 100\%$

$P_{diff}(0 \cdot deg, 0 \cdot deg) = 0\%$ $\qquad P_{diff}(30 \cdot deg, 30 \cdot deg) = 0\%$ $\qquad P_{diff}(90 \cdot deg, 90 \cdot deg) = 0\%$

These results appear to support the notion that the linear polarization states of the photons are "elements of reality." In other words, they are photon properties that exist independent of observation. However, this position is not supported by further calculation and experimentation.

Perfect anti-correlation occurs when the relative angle between the PAs is 90 degrees.

$P_{same}(0 \cdot deg, 90 \cdot deg) = 0\%$ $\qquad P_{diff}(0 \cdot deg, 90 \cdot deg) = 100\%$

At 45 degrees there is no correlation between the detectors.

$P_{same}(0 \cdot deg, 45 \cdot deg) = 50\%$ $\qquad P_{diff}(0 \cdot deg, 45 \cdot deg) = 50\%$

Using 0 degrees for both PAs as the bench mark, Herbert's analysis proceeds by moving PA_2 to 30 degrees and noting that this leads to a 25% (1 in 4) discrepancy between the analyzers.

$P_{same}(0 \cdot deg, 30 \cdot deg) = 75\%$ $\qquad P_{diff}(0 \cdot deg, 30 \cdot deg) = 25\%$

If instead PA_1 had been moved to -30 degrees the result is the same, the PAs disagree 25% of the time.

$$P_{same}(-30\cdot deg, 0\cdot deg) = 75\% \qquad P_{diff}(-30\cdot deg, 0\cdot deg) = 25\%$$

Now the locality principal is invoked. The PAs are spatially separated so that according to conventional intuition, the change in the orientation of PA_2 has no effect on the results at PA_1, and vice versa.

Now Herbert moves PA_2 back to 30 degrees with the following result.

$$P_{same}(-30\cdot deg, 30\cdot deg) = 25\% \qquad P_{diff}(-30\cdot deg, 30\cdot deg) = 75\%$$

The angular difference is now 60 degrees, and the PAs disagree 75% of the time. On the basis of local realism one would expect a discrepancy of no more than 50% . If the measurements at the PAs are independent of each other, we should simply be able to add 25% and 25%.

The experiment was performed by John Clauser and Stuart Freedman at Berkeley in 1972 and confirmed the quantum predictions. The agreement between quantum theory and experiment requires that some element of local realism must be abandoned. The consensus is that nature allows non-local interactions for entangled systems such as the photons in this example. The results at PA_1 and PA_2 (light years apart) are connected by a non-local interaction. This type of interaction is, in the words of Herbert, "unmediated, unmitigated and immediate."

Many other experiments besides those of Clauser and Freedman (most notably by Aspect and co-workers) have confirmed quantum mechanical predictions and refuted local realism. Anton Zeilinger described the current situation as follows:

> By now, a number of experiments have confirmed quantum predictions to such an extent that a local-realistic world view can no longer be maintained.
>
> It appears that, certainly at least for entangled quantum systems, it is wrong to assume that the features of the world which we observe, the measurement results, exist prior to and independently of our observation.

Appendix

In vector notation the left- and right-circular polarization states are expressed as follows:

Left circular polarization: $L = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\ i\end{pmatrix}$ Right circular polarization: $R = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\ -i\end{pmatrix}$

In tensor notation the initial photon state is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A|L\rangle_B + |R\rangle_A|R\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\ i\end{pmatrix}_A \otimes \begin{pmatrix}1\\ i\end{pmatrix}_B + \begin{pmatrix}1\\ -i\end{pmatrix}_A \otimes \begin{pmatrix}1\\ -i\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\ i\\ i\\ -1\end{pmatrix} + \begin{pmatrix}1\\ -i\\ -i\\ -1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ -1\end{pmatrix}$$

Vertical polarization: $V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Horizontal polarization: $H = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

It is easy to show that the equivalent vertical/horizontal polarization state is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_A|V\rangle_B - |H\rangle_A|H\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Naturally there are other ways to do this. The most direct would be to write |L> and |R> as superpositions of |V> and |H>, and substitute them into the initial state involving the circular polarization states.

$$\psi = \frac{1}{\sqrt{2}}\cdot(L_A \cdot L_B + R_A \cdot R_B) \left| \begin{array}{l} \text{substitute}, L_A = \frac{1}{\sqrt{2}}\cdot(V_A + i\cdot H_A) \\ \text{substitute}, L_B = \frac{1}{\sqrt{2}}\cdot(V_B + i\cdot H_B) \\ \text{substitute}, R_A = \frac{1}{\sqrt{2}}\cdot(V_A - i\cdot H_A) \\ \text{substitute}, R_B = \frac{1}{\sqrt{2}}\cdot(V_B - i\cdot H_B) \\ \text{simplify} \end{array} \right. \rightarrow \psi = \frac{-1}{2}\cdot 2^{\frac{1}{2}}\cdot\left[(-V_A)\cdot V_B + H_A\cdot H_B\right]$$