Quantum Teleportation at a Glance

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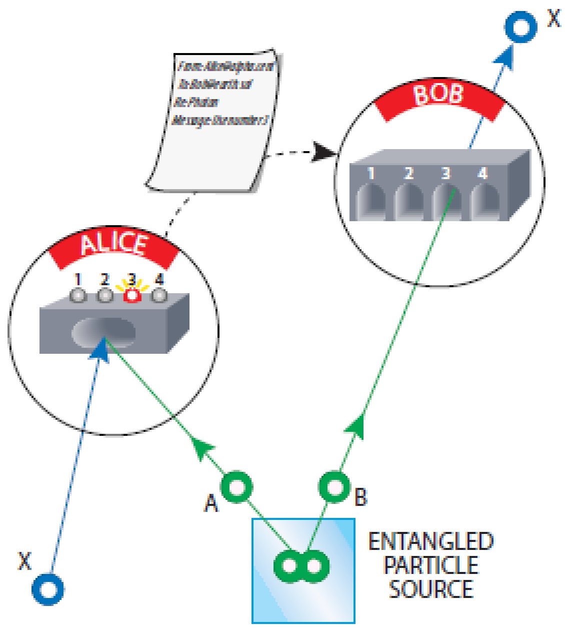
The purpose of this tutorial is to provide a brief mathematical outline of the basic elements of quantum teleportation, as illustrated in the figure below, using matrix and tensor algebra.



Alice wishes to teleport the following state (X in the figure) to Bob,

$$|\Phi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

They prepare the following entangled two-particle state, involving A and B, in which Alice has particle A and Bob has B.

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Alice arranges for the particle to be teleported, $|\Phi\rangle$, and her entangled particle to meet simultaneously on opposite sides of a beam splitter, creating the following the three-particle state.

$$|\Phi\rangle|\Psi_{AB}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ b \end{pmatrix}$$

It is not difficult to show that this state can be written as a superposition of the following 4-vectors, which are the well-known Bell states. Please see the Appendix for a definition of the Bell states.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} a \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ b \end{pmatrix} = \frac{1}{2\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -b \\ a \end{pmatrix} \right]$$

We now write this three-particle state in terms of the Bell basis labels.

$$|\Phi\rangle|\Psi_{AB}\rangle = \frac{1}{2}\left[|\Phi^+\rangle \begin{pmatrix} a \\ b \end{pmatrix} + |\Phi^-\rangle \begin{pmatrix} a \\ -b \end{pmatrix} + |\Psi^+\rangle \begin{pmatrix} b \\ a \end{pmatrix} + |\Psi^-\rangle \begin{pmatrix} -b \\ a \end{pmatrix} \right]$$

Next, Alice makes a Bell state measurement on her two particles, getting any of the four possible outcomes (Φ^+, Φ^-, Ψ^+, or Ψ^-) with equal probability, 25%. Her measurement collapses the state of Bob's particle into the companion of the result of her Bell state measurement. Alice then sends the result of her measurement through a classical channel to Bob. Depending on her report, he carries out one of the following operations on his particle to complete the teleportation process.

$$|\Phi^+\rangle \rightarrow \hat{I}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\Phi^-\rangle \rightarrow \widehat{\sigma_z}\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|\Psi^+\rangle \rightarrow \widehat{\sigma_x}\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\Psi^-\rangle \rightarrow \widehat{\sigma_z}\widehat{\sigma_x}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Appendix

The Bell basis is the following collection of maximally entangled two-qubit states.

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$