The Discrete or Quantum Fourier Transform
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The continuous-variable Fourier transforms involving position and momentum are well known. In Dirac notation (see chapter 6 in *A Modern Approach to Quantum Mechanics* by John S. Townsend) they are,

\[
\langle p | \Psi \rangle = \int \langle p | x \rangle \langle x | \Psi \rangle \, dx \quad \text{and} \quad \langle x | \Psi \rangle = \int \langle x | p \rangle \langle p | \Psi \rangle \, dp
\]

where

\[
\langle x | p \rangle = \langle p | x \rangle^* = \frac{1}{\sqrt{2\pi \hbar}} \exp \left( i \frac{2\pi px}{\hbar} \right) = \frac{1}{\sqrt{2\pi \hbar}} \exp \left( i \frac{px}{\hbar} \right)
\]

Using the coordinate and momentum completeness relations

\[
\int \langle x | x \rangle \, dx = 1 \quad \text{and} \quad \int \langle p | p \rangle \, dp = 1
\]

we can write the following generic Fourier transforms.

\[
\langle p \rangle = \int \langle p | x \rangle \langle x | \rangle \, dx \quad \text{and} \quad \langle x \rangle = \int \langle x | p \rangle \langle p | \rangle \, dp
\]

By analogy a discrete Fourier transform between the k and j indices can be created.

\[
\langle k \rangle = \sum_{j=0}^{N-1} \langle k | j \rangle \langle j \rangle
\]

were, again, by analogy

\[
\langle k | j \rangle = \frac{1}{\sqrt{N}} \exp \left( i \frac{2\pi k \cdot j}{N} \right)
\]

so that

\[
\langle k \rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp \left( i \frac{2\pi k \cdot j}{N} \right) \langle j \rangle
\]

Summing over the k index and projecting on to \( | \Psi \rangle \) yields a system of linear equations.

\[
\sum_{k=0}^{N-1} \langle k | \Psi \rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \exp \left( i \frac{2\pi k \cdot j}{N} \right) \langle j | \Psi \rangle
\]

Like all such systems it is expressible in matrix form. For example, with \( N = 2 \) and \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) as the operand we have,
Here the matrix operator is the well-known Hadamard transform. In this case it transforms spin-up in the z-direction to spin-up in the x-direction, or horizontal polarization to diagonal polarization, etc. Naturally it transforms spin-up in the x-direction to spin-up in the z-direction.

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

This, of course, also occurs with the continuous-variable Fourier transforms.

\[
\langle x | \Psi \rangle \xrightarrow{FT} \langle p | \Psi \rangle \xrightarrow{FT} \langle x | \Psi \rangle
\]

The Mathcad implementation of the discrete or quantum Fourier transform (QFT) is now demonstrated.

\[
N := 2 \quad m := 0..N-1 \quad n := 0..N-1 \quad \text{QFT}_{m,n} := \frac{1}{\sqrt{N}} \exp \left( -i \frac{2 \pi \cdot m \cdot n}{N} \right)
\]

\[
\text{QFT} = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}
\]

\[
\text{QFT} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \text{QFT} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
\text{QFT} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad \text{QFT} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

These calculations demonstrate that the QFT is a unitary operator:

\[
\text{QFT} \cdot \text{QFT} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]