
Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of $\hbar/4\pi$ in the z-basis with spin operators in the x- and y-directions.

The z-basis eigenfunctions are:

$$Sz_{\text{up}} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Sz_{\text{down}} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The x- and y-direction spin operators in the z-basis are the Pauli matrices:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The initial spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Psi := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The Appendix shows how to carry out vector tensor products in Mathcad.

The following operators represent the actual measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_x^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The three operators in tensor format are formed as follows.
\[
\sigma_{xyy} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y, \sigma_y)) \quad \sigma_{xyy} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\sigma_{xyy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_x, \sigma_y)) \quad \sigma_{xyy} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\sigma_{xyy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y, \sigma_x)) \quad \sigma_{xyy} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

That the initial state is an eigenfunction of these operators with eigenvalue +1 is now demonstrated.

\[
\Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1 \quad \Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1 \quad \Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1
\]

The fact that the operators commute means that they can have simultaneous eigenvalues.

\[
\sigma_{xyy} \cdot \sigma_{xyy} - \sigma_{xyy} \cdot \sigma_{xyy} \rightarrow 0 \quad \sigma_{xyy} \cdot \sigma_{xyy} - \sigma_{xyy} \cdot \sigma_{xyy} \rightarrow 0 \quad \sigma_{xyy} \cdot \sigma_{xyy} - \sigma_{xyy} \cdot \sigma_{xyy} \rightarrow 0
\]

The significance of this is evident when we consider the eigenvalue of the product of the three operators which obviously must be +1.

\[
(\sigma_x^1 \sigma_y^2 \sigma_y^3)(\sigma_x^1 \sigma_y^2 \sigma_y^3)(\sigma_x^1 \sigma_y^2 \sigma_x^3) = 1 \quad \Psi^T \cdot \sigma_{xyy} \cdot \sigma_{xyy} \cdot \Psi = 1
\]

If it is assumed that this result occurs because the particles are in well-defined spin states \((s_x, s_y)\) prior to measurement the following must be accepted,
\((s_x^1s_x^2s_x^3)(s_x^1s_x^2s_x^3)(s_x^1s_x^2s_x^3) = 1\)

Given that the spin measurement values can be +/- 1 and that the individual operators can have simultaneous eigenvalues, the following must be true,

\(s_x^1s_x^1 = s_x^2s_x^2 = s_x^3s_x^3 = 1\)

This reduces the previous equation involving the three operators to the following local realistic prediction.

\((s_x^1s_x^2s_x^3) = (\sigma_x^1\sigma_x^2\sigma_x^3) = 1\)

The disagreement between quantum mechanics and the local realistic view becomes starkly apparent when the \(\sigma_x^1\sigma_x^2\sigma_x^3\) measurement outcome is calculated.

\(\sigma_{xxx} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_x, \sigma_x))\)

\(\Psi^T \cdot \sigma_{xxx} \cdot \Psi = -1\)

Thus we see absolute disagreement between local realism and quantum mechanics. Local realism predicts an eigenvalue of +1 and quantum mechanics an eigenvalue of -1.

The validity of the reasoning above requires that \(\sigma_x^1\sigma_x^2\sigma_x^3\) commutes with the other operators, which we now demonstrate.

\(\sigma_{xyy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{xyy} \rightarrow 0 \quad \sigma_{yxy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yxy} \rightarrow 0 \quad \sigma_{yyx} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yyx} \rightarrow 0\)

Appendix

The tensor product of three vectors is shown below.

\(\begin{pmatrix} ace \\ acf \\ ade \end{pmatrix} \otimes \begin{pmatrix} ce \\ cf \\ de \end{pmatrix} = \begin{pmatrix} ace \\ acf \\ ade \\ bce \\ bcf \\ bde \\ bdf \end{pmatrix} \otimes \begin{pmatrix} ce \\ cf \\ de \end{pmatrix} = \begin{pmatrix} ace \\ acf \\ ade \\ bce \\ bcf \\ bde \\ bdf \end{pmatrix}\)

Mathcad does not have a command for the vector tensor product, so it is necessary to develop a way of implementing it using \text{kronecker}, which requires square matrices. For this reason the spin vector is stored in the left column of a 2x2 matrix by augmenting the spin vector with the null vector. After all the matrix tensor products have been carried out using \text{kronecker} the final spin vector resides in the left column of the final square matrix. Next the \text{submatrix} command is used to save this column, discarding the rest of the matrix.
The Mathcad syntax for the tensor multiplication of three vectors is as follows.

$$\Psi(a, b, c) := \text{submatrix} \left[ \text{kronecker} \left[ \text{augment} \left[ a, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{kronecker} \left[ \text{augment} \left[ b, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{augment} \left[ c, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \right] \right], 1, 8, 1, 1 \right]$$

The initial spin state:

$$\frac{1}{\sqrt{2}} \left( \Psi(S_{z\text{up}}, S_{z\text{up}}, S_{z\text{up}}) - \Psi(S_{z\text{down}}, S_{z\text{down}}, S_{z\text{down}}) \right) = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}$$