Elements of Reality

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Twenty years ago N. David Mermin published two articles (*Physics Today*, June 1990; *American Journal of Physics*, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (*American Journal of Physics*, December 1990; *Nature*, 3 February 2000) gedanken experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality. In what follows I present Mermin's *gedanken* experiment using tensor algebra.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of $h/4\pi$ in the z-basis with spin operators in the x- and y-directions.

The z-basis eigenfunctions are: $\quad Sz_{up} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Sz_{down} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The x- and y-direction spin operators in the z-basis are the Pauli matirces: $\quad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The initial spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The **Appendix** shows how to carry out vector tensor products in Mathcad.

The following operators represent the actual measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The three operators in tensor format are formed as follows.

$$\sigma_{xyy} := \mathrm{kronecker}\left(\sigma_x, \mathrm{kronecker}\left(\sigma_y, \sigma_y\right)\right) \qquad \sigma_{xyy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yxy} := \mathrm{kronecker}\left(\sigma_y, \mathrm{kronecker}\left(\sigma_x, \sigma_y\right)\right) \qquad \sigma_{yxy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yyx} := \mathrm{kronecker}\left(\sigma_y, \mathrm{kronecker}\left(\sigma_y, \sigma_x\right)\right) \qquad \sigma_{yyx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

That the initial state is an eigenfunction of these operators with eigenvalue +1 is now demonstrated.

$$\Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1 \qquad \Psi^T \cdot \sigma_{yxy} \cdot \Psi = 1 \qquad \Psi^T \cdot \sigma_{yyx} \cdot \Psi = 1$$

The fact that the operators commute means that they can have simultaneous eigenvalues.

$$\sigma_{xyy} \cdot \sigma_{yxy} - \sigma_{yxy} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{xyy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{yxy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{yxy} \rightarrow 0$$

The significance of this is evident when we consider the eigenvalue of the product of the three operators which obviously must be +1.

$$\left(\sigma_x^1 \sigma_y^2 \sigma_y^3\right)\left(\sigma_y^1 \sigma_x^2 \sigma_y^3\right)\left(\sigma_y^1 \sigma_y^2 \sigma_x^3\right) = 1 \qquad \Psi^T \cdot \sigma_{xyy} \cdot \sigma_{yxy} \cdot \sigma_{yyx} \cdot \Psi = 1$$

If it is assumed that this result occurs because the particles are in well-defined spin states (s_x, s_y) prior to measurement the following must be accepted,

$$\left(s_x^1 s_y^2 s_y^3\right)\left(s_y^1 s_x^2 s_y^3\right)\left(s_y^1 s_y^2 s_x^3\right)=1$$

Given that the spin measurement values can be +/- 1 and that the individual operators can have simultaneous eigenvalues, the following must be true,

$$s_y^1 s_y^1 = s_y^2 s_y^2 = s_y^3 s_y^3 = 1$$

This reduces the previous equation involving the three operators to the following local realistic prediction.

$$\left(s_x^1 s_x^2 s_x^3\right) = \left(\sigma_x^1 \sigma_x^2 \sigma_x^3\right) = 1$$

The disagreement between quantum mechanics and the local realistic view becomes starkly apparent when the $\sigma_x^1 \sigma_x^2 \sigma_x^3$ measurement outcome is calculated.

$$\sigma_{xxx} := \text{kronecker}\left(\sigma_x, \text{kronecker}\left(\sigma_x, \sigma_x\right)\right) \qquad \Psi^T \cdot \sigma_{xxx} \cdot \Psi = -1$$

Thus we see absolute disagreement between local realism and quantum mechanics. Local realism predicts an eigenvalue of +1 and quantum mechanics an eigenvalue of -1.

The validity of the reasoning above requires that $\sigma_x^1 \sigma_x^2 \sigma_x^3$ commutes with the other operators, which we now demonstrate.

$$\sigma_{xyy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{yxy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yxy} \rightarrow 0 \qquad \sigma_{yyx} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yyx} \rightarrow 0$$

Appendix

The tensor product of three vectors is shown below.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \otimes \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} ce \\ cf \\ de \\ df \end{pmatrix} = \begin{pmatrix} ace \\ acf \\ ade \\ adf \\ bce \\ bcf \\ bde \\ bdf \end{pmatrix}$$

Mathcad does not have a command for the vector tensor product, so it is necessary to develop a way of implementing it using *kronecker*, which requires square matrices. For this reason the spin vector is stored in the left column of a 2x2 matrix by augmenting the spin vector with the null vector. After all the matrix tensor products have been carried out using *kronecker* the final spin vector resides in the left column of the final square matrix. Next the *submatrix* cammand is used to save this column, discarding the rest of the matrix.

The Mathcad syntax for the tensor multiplication of three vectors is as follows.

$$\Psi(a,b,c) := \text{submatrix}\left[\text{kronecker}\left[\text{augment}\left[a,\begin{pmatrix}0\\0\end{pmatrix}\right],\text{kronecker}\left[\text{augment}\left[b,\begin{pmatrix}0\\0\end{pmatrix}\right],\text{augment}\left[c,\begin{pmatrix}0\\0\end{pmatrix}\right]\right]\right],1,8,1,1\right]$$

The initial spin state:

$$\frac{1}{\sqrt{2}}\cdot\left(\Psi\left(Sz_{up},Sz_{up},Sz_{up}\right)-\Psi\left(Sz_{down},Sz_{down},Sz_{down}\right)\right)=\begin{pmatrix}0.707\\0\\0\\0\\0\\0\\0\\-0.707\end{pmatrix}$$