Another Simulation of a GHZ Gedanken Experiment
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Many years ago N. David Mermin published two articles (Physics Today, June 1990; American Journal of
Physics, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (American Journal
of Physics, December 1990; Nature, 3 February 2000) thought experiment involving spins that sharply
revealed the clash between local realism and the quantum view of reality.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent
spin measurements will be carried out in units of \( h/4\pi \) with spin operators in the x- and y-directions.

The z-basis eigenfunctions are:

\[ S_{z\text{up}} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad S_{z\text{down}} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

The x- and y-direction spin operators:

\[ \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

The initial entangled spin state for the three spin-1/2 particles in tensor notation is:

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \]

The following operators represent the measurements to be carried out on spins 1, 2 and 3, in that order.

\[ \sigma_x \otimes \sigma_y \otimes \sigma_z \quad \sigma_y \otimes \sigma_z \otimes \sigma_y \quad \sigma_z \otimes \sigma_y \otimes \sigma_x \quad \sigma_x \otimes \sigma_z \otimes \sigma_y \]

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The four
operators in tensor format are formed as follows.

\[ \sigma_{xxy} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y, \sigma_y)) \]

\[ \sigma_{yyy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y, \sigma_y)) \]

These composite operators are Hermitian and mutually commute which means they can have simultaneous
eigenvalues.

\[ \sigma_{xxy} \sigma_{xxy} - \sigma_{xxy} \sigma_{xxy} \rightarrow 0 \quad \sigma_{xxy} \sigma_{xxy} - \sigma_{xxy} \sigma_{xxy} \rightarrow 0 \quad \sigma_{xxy} \sigma_{xxy} - \sigma_{xxy} \sigma_{xxy} \rightarrow 0 \]

The expectation values of the operators are now calculated.
\[
\Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1 \quad \Psi^T \cdot \sigma_{yxy} \cdot \Psi = 1 \quad \Psi^T \cdot \sigma_{yyx} \cdot \Psi = 1 \quad \Psi^T \cdot \sigma_{xxx} \cdot \Psi = -1
\]

Consequently the product of the four operators has the expectation value of -1.

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result as demonstrated by Mermin (Physics Today, June 1990). Looking again at the measurement operators, notice that there is a \( \sigma_x \) measurement on the first spin in the first and fourth experiment. If the spin state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1.

\[
\left( \sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_y \right) \left( \sigma^1_y \otimes \sigma^2_x \otimes \sigma^3_x \right) \left( \sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_x \right) \]

Likewise there is a \( \sigma_y \) measurement on the second spin in experiments one and three. By similar arguments those results will lead to a product of +1 also. Continuing with all pairs in the total operator using local realistic reasoning unambiguously shows that its expectation value should be +1, in sharp disagreement with the quantum mechanical result of -1. This result should cause all mathematically literate local realists to renounce and recant their heresy. However, they may resist saying this is just a thought experiment. It hasn’t actually been performed. However, if you believe in quantum simulation it has been performed.

**Quantum Simulation**

"Quantum simulation is a process in which a quantum computer simulates another quantum system. Because of the various types of quantum weirdness, classical computers can simulate quantum systems only in a clunky, inefficient way. But because a quantum computer is itself a quantum system, capable of exhibiting the full repertoire of quantum weirdness, it can efficiently simulate other quantum systems. The resulting simulation can be so accurate that the behavior the computer will be indistinguishable from the behavior of the simulated system itself." (Seth Lloyd, Programming the Universe, page 149.)

The thought experiment can be simulated using the quantum circuit shown below which is an adaptation of one that can be found at: arXiv:1712.05642v2.

\[
|1\rangle \xrightarrow{H \cdot \ldots \cdot H} \text{Measure, 0 or 1}
\]
\[
|0\rangle \xrightarrow{\ldots \oplus \cdot S \cdot H} \text{Measure, 0 or 1}
\]

The matrix operators required for the implementation of the quantum circuit:

\[
I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad \text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[
\text{HII} := \text{kronecker}(H, \text{kronecker}(I, I)) \quad \text{CNOTI} := \text{kronecker}(\text{CNOT}, I) \quad \text{ICNOT} := \text{kronecker}(I, \text{CNOT})
\]

\[
\text{ISS} := \text{kronecker}(I, \text{kronecker}(S, S)) \quad \text{SIS} := \text{kronecker}(S, \text{kronecker}(I, S)) \quad \text{SSI} := \text{kronecker}(S, \text{kronecker}(S, I))
\]

\[
\text{HHH} := \text{kronecker}(H, \text{kronecker}(H, H))
\]
First it is demonstrated that the first three steps of the circuit create the initial state.

\[
\begin{bmatrix}
\text{ICNOT-CNOTI-HII-(0 0 0 0 1 0 0 0)}
\end{bmatrix}^T = (0.707 0 0 0 0 0 -0.707)
\]

The complete circuit shown above simulates the expectation value of the \( \sigma_x \sigma_y \sigma_y \) operator. The presence of S on a line before the final H gates indicates the measurement of the \( \sigma_y \), its absence a measurement of \( \sigma_x \). The subsequent simulations show the absence of S on the middle and last line, and finally on all three lines for the simulation of the expectation value for \( \sigma_x \sigma_x \sigma_x \).

Eigenvalue \( |0\rangle = +1 \); eigenvalue \( |1\rangle = -1 \)

\[
\begin{bmatrix}
\text{HHH-SSS-ICNOT-CNOTI-HIII-(0 0 0 0 1 0 0 0 0)}
\end{bmatrix}^T = (0.5 0 0 0.5 0 0.5 0.5 0)
\]

\[
\frac{1}{2} \left( |000\rangle + |011\rangle + |101\rangle + |110\rangle \right) \Rightarrow \langle \sigma_x \sigma_y \sigma_y \rangle = 1
\]

Given the eigenvalue assignments above the expectation value associated with this measurement outcome is \( 1/4 \left[ (1)(1)+(-1)(-1)+(-1)(1)(-1)+(-1)(-1)(1) \right] = 1 \). Note that 1/2 is the probability amplitude for the product state. Therefore the probability of each member of the superposition being observed is 1/4. The same reasoning is used for the remaining simulations.

\[
\begin{bmatrix}
\text{HHH-SIS-ICNOT-CNOTI-HIII-(0 0 0 0 1 0 0 0 0)}
\end{bmatrix}^T = (0.5 0 0 0.5 0 0.5 0.5 0)
\]

\[
\frac{1}{2} \left( |000\rangle + |011\rangle + |101\rangle + |110\rangle \right) \Rightarrow \langle \sigma_y \sigma_x \sigma_y \rangle = 1
\]

\[
\begin{bmatrix}
\text{HHH-SSI-ICNOT-CNOTI-HIII-(0 0 0 0 1 0 0 0 0)}
\end{bmatrix}^T = (0.5 0 0 0.5 0 0.5 0.5 0)
\]

\[
\frac{1}{2} \left( |000\rangle + |011\rangle + |101\rangle + |110\rangle \right) \Rightarrow \langle \sigma_y \sigma_y \sigma_x \rangle = 1
\]

\[
\begin{bmatrix}
\text{HHH-ICNOT-CNOTI-HIII-(0 0 0 0 1 0 0 0)}
\end{bmatrix}^T = (0.5 0 0.5 0 0 0.5 0 0.5)
\]

\[
\frac{1}{2} \left( |001\rangle + |010\rangle + |100\rangle + |111\rangle \right) \Rightarrow \langle \sigma_y \sigma_y \sigma_y \rangle = -1
\]

Individually and in product form the simulated results are in agreement with the previous quantum mechanical calculations.

\[
\langle \sigma_x \sigma_x \sigma_x \rangle \langle \sigma_x \sigma_y \sigma_y \rangle \langle \sigma_y \sigma_y \sigma_y \rangle \langle \sigma_y \sigma_y \sigma_y \rangle = -1
\]

The appendix provides algebraic calculations of \( \langle \sigma_x \sigma_y \sigma_y \rangle \) and \( \langle \sigma_x \sigma_x \sigma_x \rangle \).
Appendix

Truth tables for the operation of the circuit elements:

\[ I = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 0 & \frac{(0 + 1)}{\sqrt{2}} \\ 1 & \frac{(0 - 1)}{\sqrt{2}} \end{pmatrix} \quad \text{CNOT} = \begin{pmatrix} 00 & 00 \\ 01 & 01 \\ 10 & 10 \\ 11 & 11 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix} \]

\[ |100\rangle \quad H \otimes I \otimes I \]
\[ \frac{1}{\sqrt{2}} \left[ |000\rangle - |100\rangle \right] \quad \text{CNOT} \otimes I \]
\[ \frac{1}{\sqrt{2}} \left[ |000\rangle - |110\rangle \right] \quad I \otimes \text{CNOT} \]
\[ \frac{1}{\sqrt{2}} \left[ |000\rangle - |111\rangle \right] \quad I \otimes S \otimes S \]
\[ \frac{1}{\sqrt{2}} \left[ |000\rangle - |1-i-i\rangle \right] \quad H \otimes H \otimes H \]
\[ \frac{1}{2} \left[ |000\rangle + |011\rangle + |101\rangle + |110\rangle \right] \]

\[ \langle \sigma_z \sigma_z \sigma_z \rangle = 1 \]