

Greenberger-Horne-Zeilinger (GHZ) Entanglement and Local Realism

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This tutorial summarizes experimental results on GHZ entanglement reported by Anton Zeilinger and collaborators in the 3 February 2000 issue of *Nature* (pp. 515-519). The GHZ experiment employs three-photon entanglement to provide a stunning attack on local realism.

First some definitions:

Realism - experiments yield values for properties that exist independent of experimental observation

Locality - the experimental results obtained at location *A* at time *t*, do not depend on the results at some other location *B* at time *t*.

H/V = horizontal/vertical linear polarization. **R/L** = right/left circular polarization. **H'/V'** rotated by 45° with respect to **H/V**.

Next some relationships between the various photon polarization states: See the appendix for vector definitions of $|H\rangle$, $|V\rangle$, $|H'\rangle$, $|V'\rangle$, $|R\rangle$ and $|L\rangle$.

$$H' = \frac{1}{\sqrt{2}} \cdot (H + V) \quad V' = \frac{1}{\sqrt{2}} \cdot (H - V) \quad R = \frac{1}{\sqrt{2}} \cdot (H + i \cdot V) \quad L = \frac{1}{\sqrt{2}} \cdot (H - i \cdot V) \quad (1)$$

$$H = \frac{1}{\sqrt{2}} \cdot (H' + V') \quad V = \frac{1}{\sqrt{2}} \cdot (H' - V') \quad H = \frac{1}{\sqrt{2}} \cdot (R + L) \quad V = \frac{i}{\sqrt{2}} \cdot (L - R) \quad (2)$$

The initial GHZ three-photon entangled state:
$$\Psi = \frac{1}{\sqrt{2}} \cdot (H_1 \cdot H_2 \cdot H_3 + V_1 \cdot V_2 \cdot V_3) \quad (3)$$

After preparation of the initial GHZ state (see figure 1 in the reference cited above), polarization measurements are performed on the three photons. Zeilinger and collaborators use **y** to stand for a circular polarization measurement and **x** for a linear polarization measurement. Initially they perform circular polarization measurements on two of the photons and a linear polarization measurement on the other photon. The quantum mechanically predicted results and actual experimental measurements are given below. The quantum predictions are obtained by substituting equations (2) into equation (3).

yyx - experiment

$$\Psi_{yyx} = \frac{1}{\sqrt{2}} \cdot \left[\frac{R_1 + L_1}{\sqrt{2}} \cdot \frac{R_2 + L_2}{\sqrt{2}} \cdot \frac{H'_3 + V'_3}{\sqrt{2}} + \frac{i \cdot (L_1 - R_1)}{\sqrt{2}} \cdot \frac{i \cdot (L_2 - R_2)}{\sqrt{2}} \cdot \frac{H'_3 - V'_3}{\sqrt{2}} \right]$$

on expansion yields

$$\Psi_{yyx} = \frac{1}{2} \cdot R_1 \cdot R_2 \cdot V'_3 + \frac{1}{2} \cdot R_1 \cdot L_2 \cdot H'_3 + \frac{1}{2} \cdot L_1 \cdot R_2 \cdot H'_3 + \frac{1}{2} \cdot L_1 \cdot L_2 \cdot V'_3$$

Each measurement (R/L or H'/V') has two possible outcomes so, in principle, there could be 8 possible results. However, quantum mechanics predicts that only four equally probable, $\left(\frac{1}{2}\right)^2 = 0.25$, outcomes are possible. The quantum mechanical prediction is in agreement with the experimental results shown in Figure 1 to within experimental error.

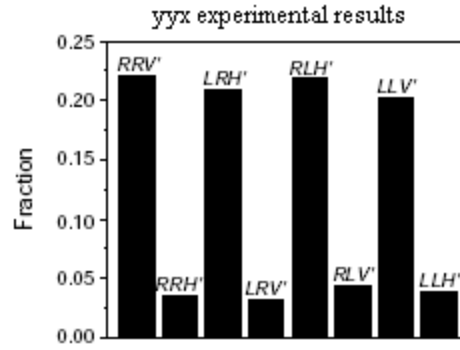


Figure 1

For the two remaining experiments in this class (**xyx** and **xyy**), the agreement between theoretical prediction and experimental results is basically the same.

While these agreements between quantum mechanics and experiment are impressive they do not directly challenge the local realist position. As will be shown later that will be accomplished by a fourth experiment involving the measurement of the linear polarization on all three photons - the **xxx** experiment.

xyx - experiment

$$\Psi_{xyx} = \frac{1}{\sqrt{2}} \left[\frac{R_1 + L_1}{\sqrt{2}} \cdot \frac{H'_2 + V'_2}{\sqrt{2}} \cdot \frac{R_3 + L_3}{\sqrt{2}} + \frac{i \cdot (L_1 - R_1)}{\sqrt{2}} \cdot \frac{H'_2 - V'_2}{\sqrt{2}} \cdot \frac{i \cdot (L_3 - R_3)}{\sqrt{2}} \right]$$

on expansion yields

$$\Psi_{xyx} = \frac{1}{2} \cdot R_1 \cdot H'_2 \cdot L_3 + \frac{1}{2} \cdot R_1 \cdot V'_2 \cdot R_3 + \frac{1}{2} \cdot L_1 \cdot H'_2 \cdot R_3 + \frac{1}{2} \cdot L_1 \cdot V'_2 \cdot L_3$$

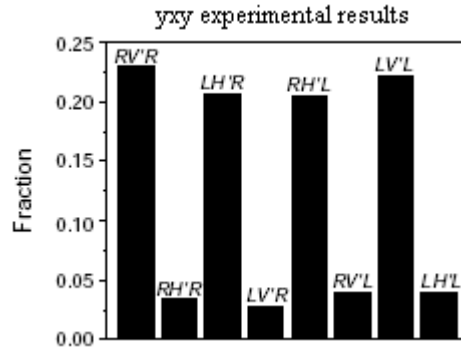


Figure 2

xyy - experiment

$$\Psi_{xyy} = \frac{1}{\sqrt{2}} \left[\frac{H'_1 + V'_1}{\sqrt{2}} \cdot \frac{R_2 + L_2}{\sqrt{2}} \cdot \frac{R_3 + L_3}{\sqrt{2}} + \frac{H'_1 - V'_1}{\sqrt{2}} \cdot \frac{i \cdot (L_2 - R_2)}{\sqrt{2}} \cdot \frac{i \cdot (L_3 - R_3)}{\sqrt{2}} \right]$$

on expansion yields

$$\Psi_{xyy} = \frac{1}{2} \cdot H'_1 \cdot R_2 \cdot L_3 + \frac{1}{2} \cdot H'_1 \cdot L_2 \cdot R_3 + \frac{1}{2} \cdot V'_1 \cdot R_2 \cdot R_3 + \frac{1}{2} \cdot V'_1 \cdot L_2 \cdot L_3$$

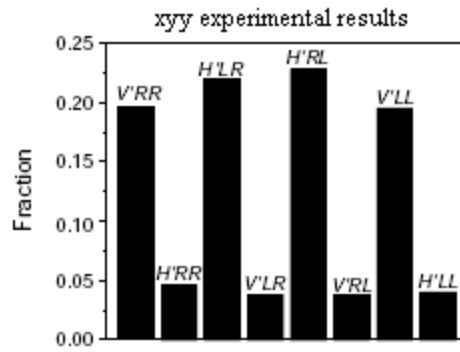


Figure 3

Now for the critical experiment.

xxx - experiment

$$\Psi_{xxx} = \frac{1}{\sqrt{2}} \left(\frac{H'_1 + V'_1}{\sqrt{2}} \cdot \frac{H'_2 + V'_2}{\sqrt{2}} \cdot \frac{H'_3 + V'_3}{\sqrt{2}} + \frac{H'_1 - V'_1}{\sqrt{2}} \cdot \frac{H'_2 - V'_2}{\sqrt{2}} \cdot \frac{H'_3 - V'_3}{\sqrt{2}} \right)$$

on expansion yields

$$\Psi_{xxx} = \frac{1}{2} \cdot H'_1 \cdot H'_2 \cdot H'_3 + \frac{1}{2} \cdot H'_1 \cdot V'_2 \cdot V'_3 + \frac{1}{2} \cdot V'_1 \cdot H'_2 \cdot V'_3 + \frac{1}{2} \cdot V'_1 \cdot V'_2 \cdot H'_3$$

This quantum mechanical prediction is displayed graphically in Figure 4.

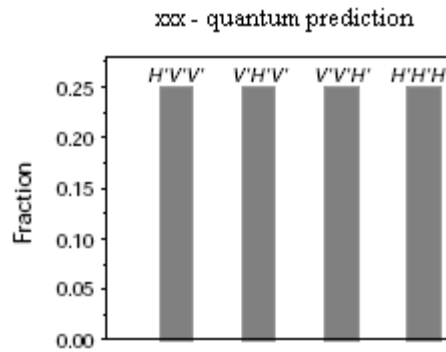


Figure 4

According to local realism the experimental outcome should be as shown in Figure 5. The origin of this prediction will be outlined shortly.

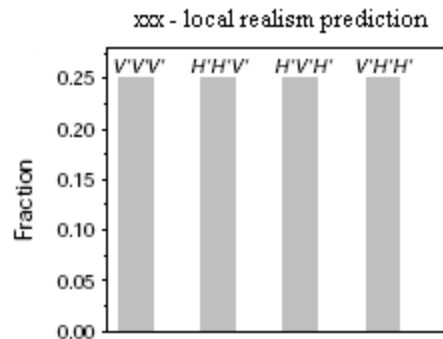


Figure 5

The quantum mechanical prediction for the xxx experiment agrees with experiment, the local realist prediction doesn't.

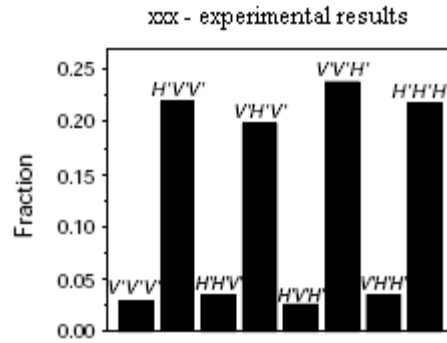


Figure 6

To explain how the local realist prediction shown in Figure 5 is derived, we recall that this position assumes that physical properties exist independent of measurement. We associate with polarization measurements the following eigenvalues: H' = +1, V' = -1, R = +1 and L = -1. Substituting these measurement eigenvalues into the first three experimental results (yyx, yxy, xyy) yields,

$$Y_1 \cdot Y_2 \cdot X_3 = Y_1 \cdot X_2 \cdot Y_3 = X_1 \cdot Y_2 \cdot Y_3 = -1$$

Therefore,

$$(Y_1 \cdot Y_2 \cdot X_3) \cdot (Y_1 \cdot X_2 \cdot Y_3) \cdot (X_1 \cdot Y_2 \cdot Y_3) = -1$$

However, this means that $X_1 \cdot X_2 \cdot X_3 = -1$ because $Y_1 \cdot Y_1 = Y_2 \cdot Y_2 = Y_3 \cdot Y_3 = 1$. According to the local realist view there are four ways to achieve this result $V_1 V_2 V_3$, $H_1 H_2 V_3$, $H_1 V_2 H_3$ and $V_1 H_2 H_3$, none of which are observed at a statistically meaningful level in the GHZ experiment.

For the xxx experiment, the mathematical predictions of quantum mechanics (Figure 4) and the local realist view (Figure 5) when compared with the actual experimental results (Figure 6) present a convincing refutation of local realism.

Appendix

$$H := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad H' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$$