Lucien Hardy's Paradox as Presented by N. David Mermin

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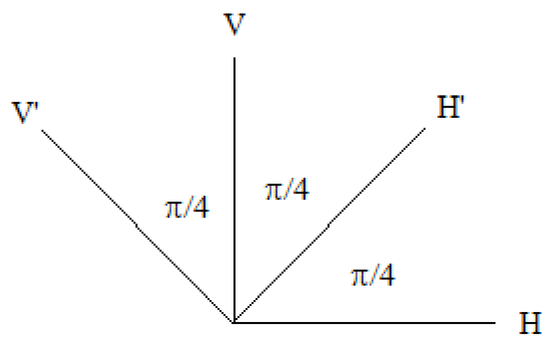
A source emits two photons in the following entangled state: $\Psi = \frac{2}{\sqrt{3}} \cdot \left(H \cdot H - \frac{1}{2} \cdot H' \cdot H' \right)$

The first photon goes to a detector to the left of the source and the second to a detector on the right.

The following diagram shows the directions that linear polarization measurements will be made on an entangled two-photon system. The detectors can be set to measure either the H-V or H'-V' mode.



Definition of polarization eigenstates:

$$H := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad H' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad V' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Ψ is expressed in tensor format: $\Psi := \frac{2}{\sqrt{3}} \cdot \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{4} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right] \qquad \Psi = \begin{pmatrix} 0.866 \\ -0.289 \\ -0.289 \\ -0.289 \end{pmatrix} \qquad \Psi^{T} \cdot \Psi = 1$

The first photon goes to Alice, the second to Bob. They record the results of a large number of independent, random polarization measurements on their photon pairs. They could for example enter numbers in the grid below which lists all the possible measurement outcomes

$$\begin{pmatrix} V'V' & V'V & V'H' & V'H \\ VV' & VV & VH' & VH \\ H'V' & H'V & H'H' & H'H \\ HV' & HV & HH' & HH \end{pmatrix}$$

The next step is to calculate the probability that these observations will be made given Ψ as the initial state. To this end we form the product state vectors in tensor format using the definitions of H, V, H' and V' provided above.

$$V'V' := \frac{1}{2} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad V'V := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad V'H' := \frac{1}{2} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \qquad V'H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$VV' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \qquad VV := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad VH' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad VH := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$H'V' := \frac{1}{2} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \qquad H'V := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad H'H' := \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad H'H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$HV' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad HV := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad HH' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad HH := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using these two-photon state functions we now calculate the probability of occurrence for each possible measurement outcome, as displayed in the following matrix.

$$\begin{bmatrix} \left(\left|V'V'^T \cdot \Psi\right|\right)^2 & \left(\left|V'V^T \cdot \Psi\right|\right)^2 & \left(\left|V'H'^T \cdot \Psi\right|\right)^2 & \left(\left|V'H^T \cdot \Psi\right|\right)^2 \\ \left(\left|VV'^T \cdot \Psi\right|\right)^2 & \left(\left|VV^T \cdot \Psi\right|\right)^2 & \left(\left|VH'^T \cdot \Psi\right|\right)^2 & \left(\left|VH^T \cdot \Psi\right|\right)^2 \\ \left(\left|H'V'^T \cdot \Psi\right|\right)^2 & \left(\left|H'V^T \cdot \Psi\right|\right)^2 & \left(\left|H'H'^T \cdot \Psi\right|\right)^2 & \left(\left|H'H^T \cdot \Psi\right|\right)^2 \\ \left(\left|HV'^T \cdot \Psi\right|\right)^2 & \left(\left|HV^T \cdot \Psi\right|\right)^2 & \left(\left|HH'^T \cdot \Psi\right|\right)^2 & \left(\left|HH^T \cdot \Psi\right|\right)^2 \end{bmatrix} = \begin{pmatrix} 0.333 & 0 & 0.333 & 0.667 \\ 0 & 0.083 & 0.167 & 0.083 \\ 0.333 & 0.167 & 0 & 0.167 \\ 0.667 & 0.083 & 0.167 & 0.75 \end{pmatrix}$$

So, where's the paradox, where's the problem? The paradox/problem is revealed by concentrating on four entries in the matrix above.

Alice	Bob	Result
V	V'	Never
V'	V	Never
V	V	Sometimes
H'	H'	Never

In any run the detectors might be set to measure [H'-V']/[H-V] or [H-V]/[H'-V']. So if one photon triggers a [H-V] detector to register V, its partner must require a [H'-V'] detector to register H'.

It follows that any [H-V]/[H-V] run in which both detectors register V (probability 0.083) each photon must require a [H'-V'] detector to register H'. Therefore, if a [H'-V']/[H'-V'] run had been selected both detectors would have registered H'.

However, the result H'H' is never observed.

Sources:

Lucien Hardy, "Spooky Action at a Distance in Quantum Mechanics," *Contemporary Physics* **39**, 419-429 (1998).

N. David Mermin, "Quantum Mysteries Refined," *American Journal of Physics* **62**, 880-887 (1994).