

Matrix Math for "Qubit Quantum Mechanics..." by Enrique Galvez, AJP 78, 510-519 (2010)

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In this document I reproduce most of the results presented in Professor Galvez's paper using the Mathcad programming environment.

State Vectors

Photon moving horizontally: $x := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Photon moving vertically: $y := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Null vector: $n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Horizontal polarization: $h := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Vertical polarization: $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Diagonal polarization: $d := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Single mode operators:

Projection operators for motion in the x- and y-directions: $X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Operator for polarizing film oriented at angle of θ to the horizontal. $\Theta_{op}(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \cdot (\cos(\theta) \quad \sin(\theta)) \rightarrow \begin{pmatrix} \cos(\theta)^2 & \cos(\theta) \cdot \sin(\theta) \\ \cos(\theta) \cdot \sin(\theta) & \sin(\theta)^2 \end{pmatrix}$

Beam splitter: $BS := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ Mirror: $M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Phase shift: $A(\delta) := \begin{pmatrix} e^{i \cdot \delta} & 0 \\ 0 & 1 \end{pmatrix}$

Half and quarter wave plate: $W_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $W_4 := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ Identity: $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotated half wave plate: $W_2(\theta) := \begin{pmatrix} \cos(2 \cdot \theta) & \sin(2 \cdot \theta) \\ \sin(2 \cdot \theta) & -\cos(2 \cdot \theta) \end{pmatrix}$ $W(\theta) := \begin{pmatrix} \cos(2 \cdot \theta) & \sin(2 \cdot \theta) & 0 & 0 \\ \sin(2 \cdot \theta) & -\cos(2 \cdot \theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Mach-Zehnder interferometer: $MZ(\delta) := BS \cdot A(\delta) \cdot M \cdot BS$

Two mode states and operators:

Single-photon direction of propagation and polarization states:

$xh := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $xv := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $yh := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $yv := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Two-photon direction of propagation states.

$$\begin{aligned}
 \text{xx} &:= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \text{xy} &:= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \text{yx} &:= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \text{yy} &:= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
 \end{aligned}$$

Two-photon polarization states.

$$\begin{aligned}
 \text{hh} &:= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \text{hv} &:= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \text{vh} &:= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \text{vv} &:= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
 \end{aligned}$$

Polarizing beam splitter which transmits horizontally polarized photons and reflects vertically polarized photons.

$$\text{PBS} := \text{xh} \cdot \text{xh}^T + \text{yv} \cdot \text{xv}^T + \text{yh} \cdot \text{yh}^T + \text{xv} \cdot \text{yv}^T \quad \text{PBS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Polarization M-Z interferometer: $\text{MZ}_P(\delta) := \text{PBS} \cdot \text{kronacker}(\text{A}(\delta), \text{I}) \cdot \text{kronacker}(\text{M}, \text{I}) \cdot \text{PBS}$

Kronecker is Mathcad's command for tensor multiplication of square matrices.

Mach-Zehnder interferometer for direction of propagation and polarization, which places a rotatable half-wave plate in the upper path.

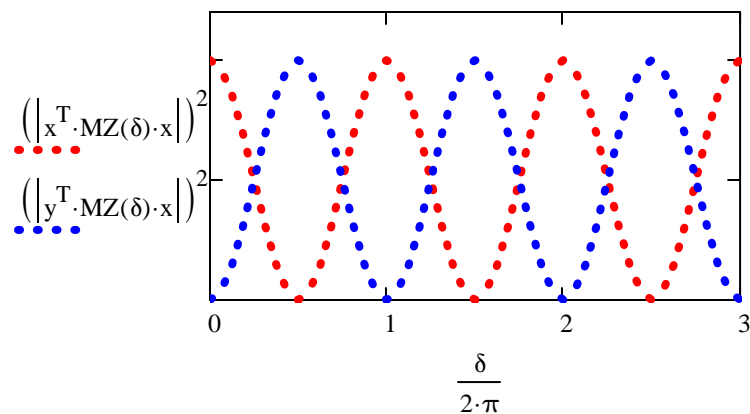
$$\text{MZ}_{dp}(\theta, \delta) := \text{kronacker}(\text{BS}, \text{I}) \cdot \text{kronacker}(\text{A}(\delta), \text{I}) \cdot \text{W}(\theta) \cdot \text{kronacker}(\text{M}, \text{I}) \cdot \text{kronacker}(\text{BS}, \text{I})$$

Mach-Zehnder two-photon direction-of-propagation interferometer.

$$\text{BSBS} := \text{kronacker}(\text{BS}, \text{BS}) \quad \text{MM} := \text{kronacker}(\text{M}, \text{M}) \quad \text{AA}(\delta) := \text{kronacker}(\text{A}(\delta), \text{A}(\delta))$$

$$\text{MZ}_{dd}(\delta) := \text{BSBS} \cdot \text{AA}(\delta) \cdot \text{MM} \cdot \text{BSBS}$$

Confirm the results in Figure 2 for the Mach-Zehnder interferometer: $\delta := 0, .125 \cdot \pi .. 6\pi$



Demonstrate that a superposition is formed after first beam splitter

$$\text{BS} \cdot x = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot (x + i \cdot y) = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

Confirmation that path information destroys interference. $\delta := 0, .2 \cdot \pi \dots \pi$

x-direction: $\theta = 0$, no path information

$\theta = \pi/4$, path information

$$\left(\left| \text{kronacker}(X, I) \cdot \text{MZ}_{\text{dp}}(0, \delta) \cdot x_v \right| \right)^2 =$$

1
0.905
0.655
0.345
0.095
0

$$\left(\left| \text{kronacker}(Y, I) \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot x_v \right| \right)^2 =$$

0.5
0.5
0.5
0.5
0.5
0.5
0.5

y-direction: $\theta = 0$, no path information

$\theta = \pi/4$, path information

$$\left(\left| \text{kronacker}(Y, I) \cdot \text{MZ}_{\text{dp}}(0, \delta) \cdot x_v \right| \right)^2 =$$

0
0.095
0.345
0.655
0.905
1

$$\left(\left| \text{kronacker}(Y, I) \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot x_v \right| \right)^2 =$$

0.5
0.5
0.5
0.5
0.5
0.5
0.5

Erasure of path information restores interference. Erasers for the x- and y-directions place diagonal polarizers in those directions after the interferometer.

$$E_x := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_y := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The x-direction has an eraser and the y-direction does not.

x-direction:

$$\left(\left| \text{kroncker}(X, I) \cdot E_x \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot \text{xv} \right|^2 \right) =$$

0
0.048
0.173
0.327
0.452
0.5

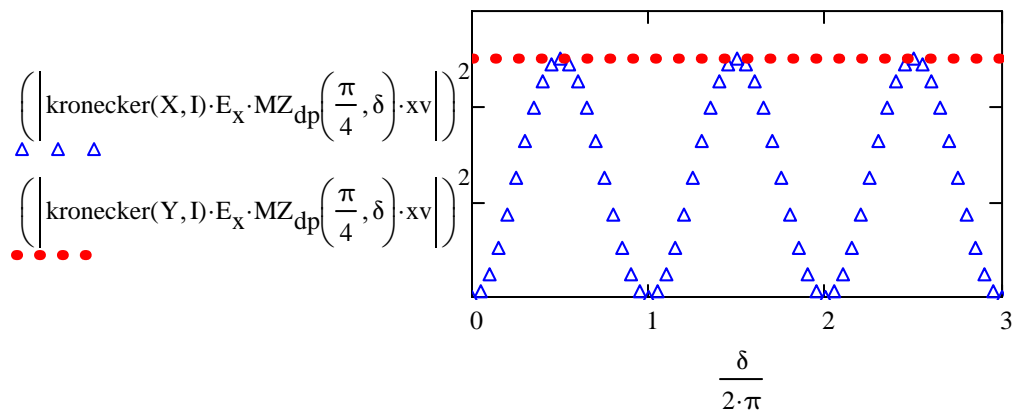
y-direction:

$$\left(\left| \text{kroncker}(Y, I) \cdot E_x \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot \text{xv} \right|^2 \right) =$$

0.5
0.5
0.5
0.5
0.5
0.5

Display Figure 3b results.

$\delta := 0, .1 \cdot \pi .. 6\pi$



The y-direction has an eraser and the x-direction does not.

$\delta := 0, .2 \cdot \pi .. \pi$

x-direction:

$$\left(\left| \text{kroncker}(X, I) \cdot E_y \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot \text{xv} \right|^2 \right) =$$

0.5
0.5
0.5
0.5
0.5
0.5

y-direction:

$$\left(\left| \text{kroncker}(Y, I) \cdot E_y \cdot \text{MZ}_{\text{dp}}\left(\frac{\pi}{4}, \delta\right) \cdot \text{xv} \right|^2 \right) =$$

0.5
0.452
0.327
0.173
0.048
0

For the MZ polarization interferometer diagonally polarized light enters in the x-direction, $|x_d\rangle$. Tensor vector multiplication is awkward in Mathcad as is shown below.

$$\Psi_{\text{in}} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{submatrix}(\text{kroncker}(\text{augment}(x, n), \text{augment}(d, n)), 1, 4, 1, 1) = \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \\ 0 \end{pmatrix}$$

No light, however, exits in the x-direction. It exits in the y-direction showing no interference effects.

x-direction:

$$\left(\left| \text{kroncker}(X, I) \cdot \text{MZ}_P(\delta) \cdot \Psi_{\text{in}} \right| \right)^2 =$$

0
0
0
0
0
0

y-direction:

$$\left(\left| \text{kroncker}(Y, I) \cdot \text{MZ}_P(\delta) \cdot \Psi_{\text{in}} \right| \right)^2 =$$

1
1
1
1
1
1

Placement of a D polarizer in the y-direction output erases distinguishing information and interference appears.

$$\left(\left| \text{kroncker}(Y, I) \cdot E_y \cdot \text{MZ}_P(\delta) \cdot \Psi_{\text{in}} \right| \right)^2 =$$

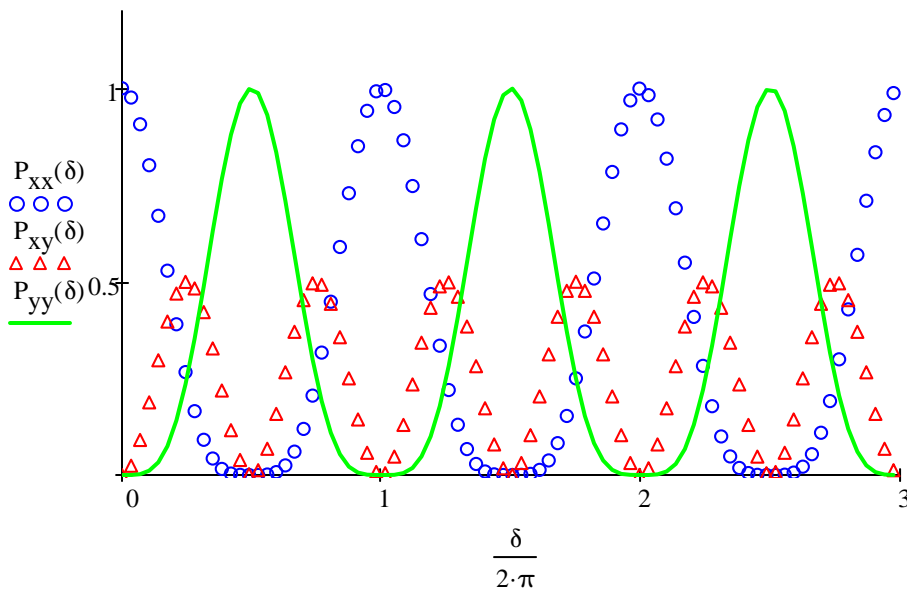
1
0.905
0.655
0.345
0.095
0

Calculation of exit probabilities for two photons in direction-of-propagation modes:

$$P_{xx}(\delta) := \left(\left| xx^T \cdot \text{MZ}_{dd}(\delta) \cdot xx \right| \right)^2 \quad P_{xy}(\delta) := \left[\left| \frac{1}{\sqrt{2}} \cdot (xy + yx)^T \cdot \text{MZ}_{dd}(\delta) \cdot xx \right| \right]^2 \quad P_{yy}(\delta) := \left(\left| yy^T \cdot \text{MZ}_{dd}(\delta) \cdot xx \right| \right)^2$$

Reproduction of Figure 5b with the addition of P_{yy} .

$$\delta := 0, .07 \cdot \pi .. 6 \cdot \pi$$



Hong-Ou-Mandel interference
(right column, page 516):

$$\text{BSBS} \cdot \frac{1}{\sqrt{2}} \cdot (xy + yx) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix} \quad \frac{i}{\sqrt{2}} \cdot (xx + yy) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix}$$

Section III.D deals with distinguishing between pure and mixed states experimentally. The pure state and its density matrix are given below.

$$\Psi_{\text{pure}} := \frac{1}{\sqrt{2}} \cdot (\text{hh} + \text{vv}) \quad \Psi_{\text{pure}} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \quad \Psi_{\text{pure}} \cdot \Psi_{\text{pure}}^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

The density matrix for the mixed state is calculated as follows.

$$\frac{1}{2} \cdot \text{hh} \cdot \text{hh}^T + \frac{1}{2} \cdot \text{vv} \cdot \text{vv}^T = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

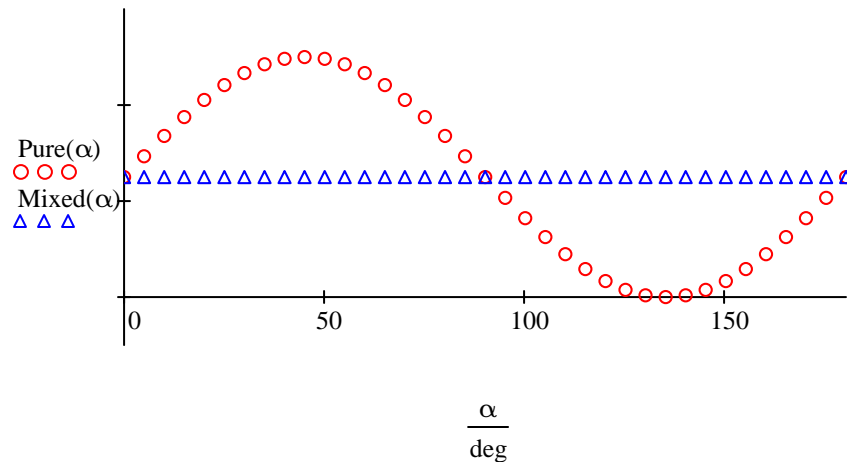
The following calculations and their graphical representation are in complete agreement with section III.D

$$\text{Pure}(\alpha) := \text{tr} \left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix} \cdot (\cos(\alpha) \quad \cos(\alpha) \quad \sin(\alpha) \quad \sin(\alpha)) \right] \text{ simplify} \rightarrow \frac{\sin(2 \cdot \alpha)}{4} + \frac{1}{4}$$

$$\text{Mixed}(\alpha) := \text{tr} \left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix} \cdot (\cos(\alpha) \quad \cos(\alpha) \quad \sin(\alpha) \quad \sin(\alpha)) \right] \text{ simplify} \rightarrow \frac{1}{4}$$

Reproduce Figure 6 results.

$\alpha := 0 \cdot \text{deg}, 5 \cdot \text{deg} .. 180 \cdot \text{deg}$



$$\text{kroncker}(W_2(0), I) \cdot \Psi_{\text{pure}} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$\text{Pure}(\alpha) := \text{tr} \left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix} \cdot (\cos(\alpha) \quad \cos(\alpha) \quad \sin(\alpha) \quad \sin(\alpha)) \right] \text{ simplify } \rightarrow \frac{1}{4} - \frac{\sin(2 \cdot \alpha)}{4}$$