Raising & Lowering; Creating & Annihilating

Frank Rioux

The purpose of this tutorial is to illustrate uses of the creation (raising) and annihilation (lowering) operators in the complementary coordinate and matrix representations. These operators have routine utility in quantum mechanics in general, and are especially useful in the areas of quantum optics and quantum information.

The harmonic oscillator eigenstates are regularly used to represent (in a rudimentary way) the vibrational states of diatomic molecules and also (more rigorously) the quantized states of the electromagnetic field. The creation operator adds a quantum of energy to the molecule or the electromagnetic field and the annihilation operator does the opposite.

The harmonic oscillator eigenfunctions in coordinate space are given below, where **v** is the quantum number and can have the values 0, 1, 2, ...

$$\Psi(v,x) := \frac{1}{\sqrt{2^{v}\cdot v!\cdot\sqrt{\pi}}}\cdot Her(v,x)\cdot\exp\left(\frac{-x^{2}}{2}\right)$$

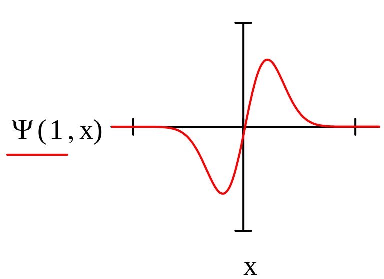
First we demonstrate that the harmonic oscillator eigenfunctions are normalized.

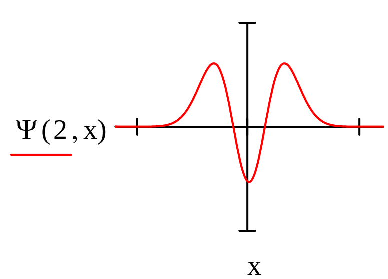
$$\int_{-\infty}^{\infty}\Psi(1,x)^{2}\,dx = 1 \qquad \int_{-\infty}^{\infty}\Psi(2,x)^{2}\,dx = 1 \qquad \int_{-\infty}^{\infty}\Psi(3,x)^{2}\,dx = 1$$

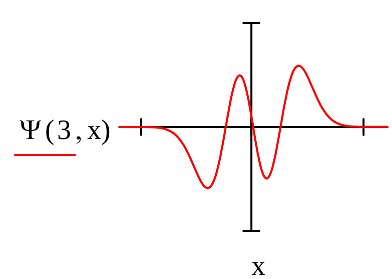
Next we demonstrate that they are orthogonal:

$$\int_{-\infty}^{\infty}\Psi(1,x)\cdot\Psi(2,x)\,dx = 0 \qquad \int_{-\infty}^{\infty}\Psi(1,x)\cdot\Psi(3,x)\,dx = 0 \qquad \int_{-\infty}^{\infty}\Psi(2,x)\cdot\Psi(3,x)\,dx = 0$$

The harmonic oscillator eigenfunctions form an orthonormal basis set. They are displayed below.



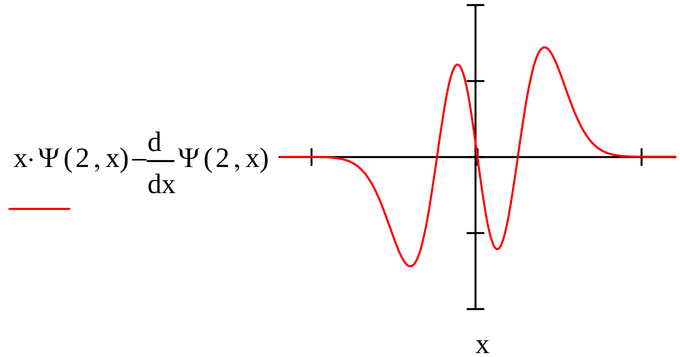




The raising or creation operator in the coordinate representation in reduced units is the position operator minus *i* times the coordinate space momentum operator:

$$x\cdot\blacksquare - \frac{d}{dx}\blacksquare$$

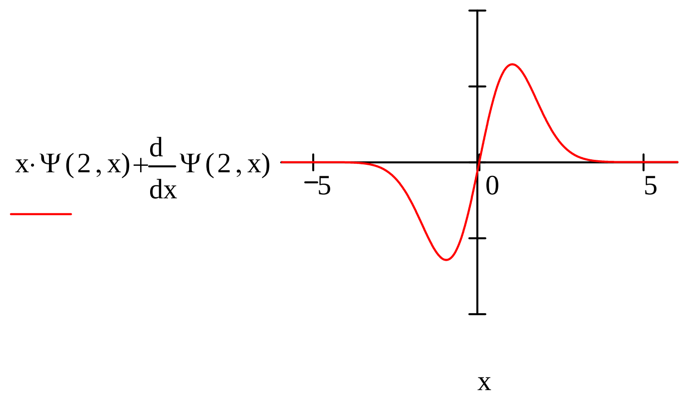
Operating on the v = 2 eigenfunction yields the v = 3 eigenfunction:



The lowering or annihilation operator in the coordinate representation in reduced units is the position operator plus *i* times the coordinate space momentum operator:

$$x\cdot\blacksquare + \frac{d}{dx}\blacksquare$$

Operating on the v = 2 eigenfunction yields the v = 1 eigenfunction:



The energy operator in coordinate space and the energy expectation value for the v = 2 state are given below. $E_v = v + 1/2$ in atomic units.

$$H = \frac{-1}{2}\cdot\frac{d^2}{dx^2}\blacksquare + \frac{1}{2}\cdot x^2\cdot\blacksquare \qquad \int_{-\infty}^{\infty}\Psi(2,x)\cdot\left(\frac{-1}{2}\cdot\frac{d^2}{dx^2}\Psi(2,x) + \frac{1}{2}\cdot x^2\cdot\Psi(2,x)\right)dx = 2.5$$

In the matrix formulation of quantum mechanics the harmonic oscillator eigenfunctions are vectors. The matrix representations for the v = 0 state and the eigenstates used above are given below. They are actually infinite vectors which for practical purposes are truncated at order 5.

$$v0 := \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \qquad v1 := \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \qquad v2 := \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \qquad v3 := \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$$

They form an orthonormal basis set:

$v1^T \cdot v1 = 1 \quad v2^T \cdot v2 = 1 \quad v3^T \cdot v3 = 1 \qquad v1^T \cdot v2 = 0 \quad v1^T \cdot v3 = 0 \quad v2^T \cdot v3 = 0$

In this context the creation and annihilation operators are 5x5 matrices.

$$\text{Create} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix} \qquad \text{Annihilate} := \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The annihilation operator on the v = 2 state: $\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$ $\quad \text{Annihilate} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1.414 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The annihilation operator on the v = 0 state: $\text{Annihilate} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The creation operator on the v = 2 state: $\hat{a}^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$ $\quad \text{Create} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1.732 \\ 0 \end{pmatrix}$

The number operator on the v = 2 state: $\hat{a}^{\dagger}\hat{a}\left|n\right\rangle = n\left|n\right\rangle$ $\quad \text{Create} \cdot \text{Annihilate} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

Or do it this way: $v0^T \cdot Create \cdot Annihilate \cdot v0 = 0$ $v1^T \cdot Create \cdot Annihilate \cdot v1 = 1$

$v2^T \cdot Create \cdot Annihilate \cdot v2 = 2$ $v3^T \cdot Create \cdot Annihilate \cdot v3 = 3$

The energy operator operating on the v = 2 and 5 states: $\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle$

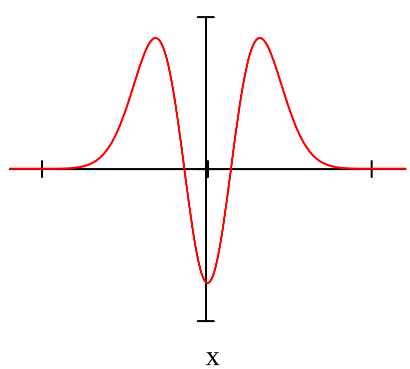
$$(Create \cdot Annihilate) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2.5 \\ 0 \\ 0 \end{pmatrix} \qquad (Create \cdot Annihilate) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4.5 \end{pmatrix}$$

Creating the v = 2 eigenstate from the vacuum: $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$ $\quad \frac{1}{\sqrt{2!}} \cdot Create^2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\frac{1}{\sqrt{2!}} \cdot Create^2 \cdot v0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This operation is illustrated graphically in the coordinate representation as follows:

$$x \cdot \left(x \cdot \Psi(0,x) - \frac{d}{dx}\Psi(0,x)\right) - \frac{d}{dx}\left(x \cdot \Psi(0,x) - \frac{d}{dx}\Psi(0,x)\right)$$



Construct the matrix forms of the position and momentum operators using the annihilation and creation operators. See E. E. Anderson, *Modern Physics and Quantum Mechanics*, page 201.

$$\mathrm{Position} := \frac{\mathrm{Annihilate} + \mathrm{Create}}{\sqrt{2}} \qquad \mathrm{Momentum} := \frac{i}{\sqrt{2}} \cdot (\mathrm{Create} - \mathrm{Annihilate})$$

Calculate the position and momentum expectation values for several states:

$$v0^T \cdot \mathrm{Position} \cdot v0 = 0 \quad v0^T \cdot \mathrm{Momentum} \cdot v0 = 0 \quad v1^T \cdot \mathrm{Position} \cdot v1 = 0 \quad v1^T \cdot \mathrm{Momentum} \cdot v1 = 0$$

Calculate the position-momentum uncertainty product (ΔxΔp) for several states:

$$\sqrt{v0^T \cdot \mathrm{Position}^2 \cdot v0 - \left(v0^T \cdot \mathrm{Position} \cdot v0\right)^2} \cdot \sqrt{v0^T \cdot \mathrm{Momentum}^2 \cdot v0 - \left(v0^T \cdot \mathrm{Momentum} \cdot v0\right)^2} = 0.5$$

$$\sqrt{v1^T \cdot \mathrm{Position}^2 \cdot v1 - \left(v1^T \cdot \mathrm{Position} \cdot v1\right)^2} \cdot \sqrt{v1^T \cdot \mathrm{Momentum}^2 \cdot v1 - \left(v1^T \cdot \mathrm{Momentum} \cdot v1\right)^2} = 1.5$$

Calculate the energy expectation value for the following superposition state.

$$\Psi = \frac{1}{\sqrt{2}} \cdot v0 + \frac{1}{\sqrt{3}} \cdot v1 + \frac{1}{\sqrt{6}} \cdot v2$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ 0 \end{pmatrix}^T \cdot \left[\mathrm{Create} \cdot \mathrm{Annihilate} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ 0 \end{pmatrix} \right] \rightarrow \frac{7}{6}$$

$$P_0 \cdot E_0 + P_1 \cdot E_1 + P_2 \cdot E_2 = \frac{7}{6} \qquad \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{2} + \frac{1}{6} \cdot \frac{5}{2} \rightarrow \frac{7}{6}$$

Below it is demonstrated that there are two equivalent forms of the harmonic oscillator energy operator in the matrix formulation of quantum mechanics.

$$\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)|n\rangle=\left(n+\frac{1}{2}\right)|n\rangle \qquad \left(\hat{a}\hat{a}^{\dagger}-\frac{1}{2}\right)|n\rangle=\left(n+\frac{1}{2}\right)|n\rangle$$

$$\text{Create}\cdot\text{Annihilate}\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\3.5\\0\end{pmatrix} \qquad \text{Annihilate}\cdot\text{Create}\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}-\frac{1}{2}\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\3.5\\0\end{pmatrix}$$

Or, do it this way:

$$\left(\frac{\text{Create}\cdot\text{Annihilate}+\text{Annihilate}\cdot\text{Create}}{2}\right)\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\3.5\\0\end{pmatrix}$$

Or this way:

$$\left(\frac{\text{Momentum}^2}{2}+\frac{\text{Position}^2}{2}\right)\cdot\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\3.5\\0\end{pmatrix}$$

Demonstrate that the position and momentum operators don't commute using the matrix form of the operators.

$$i\cdot(\text{Momentum}\cdot\text{Position}-\text{Position}\cdot\text{Momentum})=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-4\end{pmatrix}$$

This calculation yields the identity matrix as expected, except for the value of the last diagonal element. The latter is a mathematical artifact of using truncated matrices for operators which are infinite.