Silver atoms are deflected by an inhomogeneous magnetic field because of the two-valued magnetic moment associated with their unpaired 5s electron ([Kr]5s14d10). The beam of silver atoms entering the Stern-Gerlach magnet oriented in the z-direction (SGZ) on the left is unpolarized. This means it is a mixture of randomly polarized Ag atoms. A mixture cannot be represented by a wave function, it requires a density matrix, as will be shown later.

This situation is exactly analogous to the three-polarizer demonstration. Light emerging from an incandescent light bulb is unpolarized, a mixture of all possible polarization angles, so we can't write a wave function for it. The first Stern-Gerlach magnet plays the same role as the first polarizer, it forces the Ag atoms into one of measurement eigenstates - spin-up or spin-down in the z-direction. The only difference is that in the three-polarizer demonstration only one state was created - vertical polarization. Both demonstrations illustrate that the only values that are observed in an experiment are the eigenvalues of the measurement operator.

To continue with the analysis of the Stern-Gerlach demonstration we need vectors to represent the various spin states of the Ag atoms.

**Spin Eigenfunctions**

**Spin-up in the z-direction:** \( \alpha_z := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)  
**Spin-down in the z-direction:** \( \beta_z := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

**Spin-up in the x-direction:** \( \alpha_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)  
**Spin-down in the x-direction:** \( \beta_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

In the next step, the spin-up beam (deflected toward by the magnet's north pole) enters a magnet oriented in the x-direction, SGX. The \( \alpha_z \) beam splits into \( \alpha_x \) and \( \beta_x \) beams of equal intensity. This is because it is a superposition of the x-direction spin eigenstates as shown below.

\[
\frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} (\alpha_x + \beta_x) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Next the \( \alpha_x \) beam is directed toward a second SGZ magnet and splits into two equal \( \alpha_z \) and \( \beta_z \) beams. This happens because \( \alpha_x \) is a superposition of the \( \alpha_z \) and \( \beta_z \) spin states.
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \frac{1}{\sqrt{2}} (\alpha_z + \beta_z) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

Operators

We can also use the Pauli operators (in units of $\hbar/4\pi$) to analyze this experiment.

**SGZ operator:** $\text{SGZ} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

**SGX operator:** $\text{SGX} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The probability that an $\alpha_z$ Ag atom will emerge spin-up after passing through a SGX magnet:

- Probability amplitude: $\alpha_x^T \cdot \text{SGX} \cdot \alpha_z = 0.707$
- Probability: $\left( \alpha_x^T \cdot \text{SGX} \cdot \alpha_z \right)^2 = 0.5$

The probability that an $\alpha_z$ Ag atom will emerge spin-down after passing through a SGX magnet:

- Probability amplitude: $\beta_x^T \cdot \text{SGX} \cdot \alpha_z = -0.707$
- Probability: $\left( \beta_x^T \cdot \text{SGX} \cdot \alpha_z \right)^2 = 0.5$

The probability that an $\alpha_x$ Ag atom will emerge spin-up after passing through a SGZ magnet:

- Probability amplitude: $\alpha_z^T \cdot \text{SGX} \cdot \alpha_x = 0.707$
- Probability: $\left( \alpha_z^T \cdot \text{SGX} \cdot \alpha_x \right)^2 = 0.5$

The probability that an $\alpha_x$ Ag atom will emerge spin-down after passing through a SGZ magnet:

- Probability amplitude: $\beta_z^T \cdot \text{SGX} \cdot \alpha_x = 0.707$
- Probability: $\left( \beta_z^T \cdot \text{SGX} \cdot \alpha_x \right)^2 = 0.5$

In examining the figure above we note that the SGX magnet destroys the entering $\alpha_z$ state, creating a superposition of spin-up and spin-down in the x-direction. Again measurement forces the system into one of the eigenstates of the measurement operator.

**Density Operator (Matrix) Approach**

A more general analysis is based on the concept of the density operator (matrix), in general given by the following outer product $|\Psi \rangle \langle \Psi|$. It is especially important because it can be used to represent mixtures, which cannot be represented by wave functions as noted above.

For example, the probability that an $\alpha_z$ spin system will emerge in the $\alpha_x$ channel of a SGX magnet is equal to the trace of the product of the density matrices representing the $\alpha_z$ and $\alpha_x$ states as shown below.

$$\langle \alpha_x \| \alpha_z \rangle^2 = \langle \alpha_x | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle = \sum_i \langle \alpha_z | i \rangle \langle i | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle = \sum_i \langle i | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle \langle \alpha_z | i \rangle$$

$$= \text{Tr} \left( | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle \langle \alpha_z | \right) = \text{Tr} \left( \rho_{\alpha_x} \rho_{\alpha_z} \right)$$

where the completeness relation $\sum_i |i \rangle \langle i | = 1$ has been employed.
Density matrices for spin-up and spin-down in the z-direction:
\[ \rho_{\alpha z} := \alpha_z \cdot \alpha_z^T \quad \rho_{\beta z} := \beta_z \cdot \beta_z^T \]

Density matrices for spin-up and spin-down in the x-direction:
\[ \rho_{\alpha x} := \alpha_x \cdot \alpha_x^T \quad \rho_{\beta x} := \beta_x \cdot \beta_x^T \]

An unpolarized spin system can be represented by a 50-50 mixture of any two orthogonal spin density matrices. Below it is shown that using the z-direction and the x-direction give the same answer.
\[ \rho_{\text{mix}} := \frac{1}{2} \rho_{\alpha z} + \frac{1}{2} \rho_{\beta z} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \]

Now we re-analyze the Stern-Gerlach experiment using the density operator (matrix) approach.

The probability that an unpolarized spin system will emerge in the \( \alpha_z \) channel of a SGZ magnet is 0.5:
\[ \text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.5 \]

The probability that the \( \alpha_z \) beam will emerge in the \( \alpha_x \) channel of a SGX magnet is 0.5:
\[ \text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) = 0.5 \]

The probability that the \( \alpha_x \) beam will emerge in the \( \alpha_z \) channel of the final SGZ magnet is 0.5:
\[ \text{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) = 0.5 \]

The probability that the \( \alpha_x \) beam will emerge in the \( \beta_z \) channel of the final SGZ magnet is 0.5:
\[ \text{tr}(\rho_{\beta z} \cdot \rho_{\alpha x}) = 0.5 \]

After the final SGZ magnet, 1/8 of the original Ag atoms emerge in the \( \alpha_z \) channel and 1/8 in the \( \beta_z \) channel.
\[ \text{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) \cdot \text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.125 \quad \text{tr}(\rho_{\beta z} \cdot \rho_{\alpha x}) \cdot \text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.125 \]