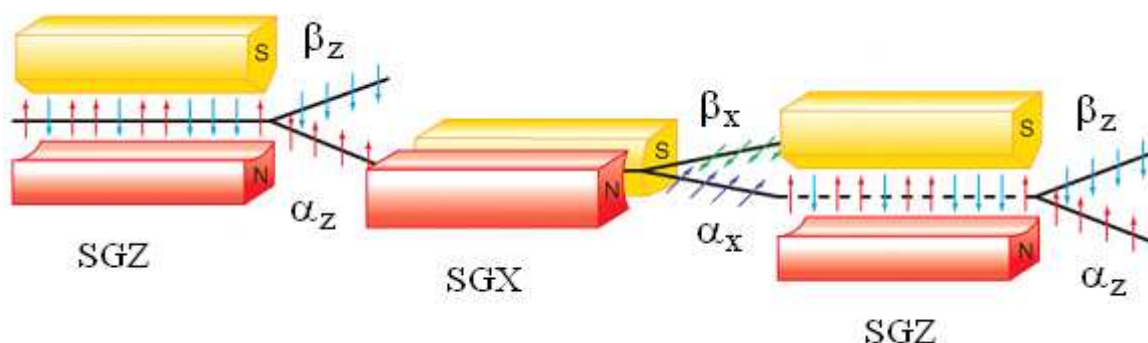


# Analysis of the Stern-Gerlach Experiment Using Matrix Mechanics

Frank Rioux  
 Department of Chemistry  
 College of Saint Benedict | Saint John's University  
 St. Joseph, MN 56374

The purpose of this tutorial is to analyze the Stern-Gerlach experiment using matrix mechanics. The figure below is taken (and modified) from Thomas Engel's text, *Quantum Chemistry & Spectroscopy*.



Silver atoms are deflected by an inhomogeneous magnetic field because of the two-valued magnetic moment associated with their unpaired 5s electron ( $[\text{Kr}]5s^14d^{10}$ ). The beam of silver atoms entering the Stern-Gerlach magnet oriented in the z-direction (SGZ) on the left is unpolarized. This means it is a mixture of randomly spin-polarized Ag atoms. As such, it is impossible to write a quantum mechanical wavefunction for this initial state.

This situation is exactly analogous to the three-polarizer demonstration described in a previous tutorial (<http://www.users.csbsju.edu/~frioux/polarize/polarize.pdf>). Light emerging from an incandescent light bulb is unpolarized, a mixture of all possible polarization angles, so we can't write a wave function for it. The first Stern-Gerlach magnet plays the same role as the first polarizer, it forces the Ag atoms into one of measurement eigenstates - spin-up or spin-down in the z-direction. The only difference is that in the three-polarizer demonstration only one state was created - vertical polarization. Both demonstrations illustrate an important quantum mechanical postulate - the only values that are observed in a measurement are the eigenvalues of the measurement operator.

To continue with the analysis of the Stern-Gerlach demonstration we need vectors to represent the various spin states of the Ag atoms. We will restrict our attention to the x- and z- spin directions, although the spin states for the y-direction are also available.

## Spin Eigenfunctions

$$\text{Spin-up in the z-direction: } \alpha_z := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Spin-down in the z-direction: } \beta_z := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Spin-up in the x-direction: } \alpha_x := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{Spin-down in the x-direction: } \beta_x := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

After the SGZ magnet, the spin-up beam (deflected toward the magnet's north pole) enters a magnet oriented in the x-direction, SGX. The  $\alpha_z$  beam splits into  $\alpha_x$  and  $\beta_x$  beams of equal intensity. This is because it is a superposition of the x-direction spin eigenstates as shown below.

$$\frac{1}{\sqrt{2}} \cdot \left[ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right] \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot (\alpha_x + \beta_x) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Next the  $\alpha_x$  beam is directed toward a second SGZ magnet and splits into two equal  $\alpha_z$  and  $\beta_z$  beams. This happens because  $\alpha_x$  is a superposition of the  $\alpha_z$  and  $\beta_z$  spin states.

$$\frac{1}{\sqrt{2}} \cdot \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \rightarrow \begin{pmatrix} \frac{1}{2} \cdot 2^{\frac{1}{2}} \\ \frac{1}{2} \cdot 2^{\frac{1}{2}} \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot (\alpha_z + \beta_z) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

Summarizing, the first SGZ magnet creates two beams in well-defined states,  $\alpha_z$  and  $\beta_z$ . In quantum mechanics this is called state preparation. Subsequent interaction of the  $\alpha_z$  state with the SGX magnet destroys that state, as can be seen at the output ports of the final SGZ magnet. A beam that was pure  $\alpha_z$  is now an even superposition of  $\alpha_z$  and  $\beta_z$ .

It should also be pointed out that the overall results for the Stern-Gerlach experiment are the same whether the beams are intense or weak, such that there is only one Ag atom in the apparatus at any time.

## Operators

We can also use operators (matrices) along with vectors to analyze this experiment. The matrix operators associated with the two Stern-Gerlach magnets are shown below.

$$\text{SGZ operator:} \quad \text{SGZ} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{SGX operator:} \quad \text{SGX} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The spin states  $\alpha_z$  and  $\beta_z$  are eigenfunctions of the SGZ operator with eigenvalues +1 and -1, respectively:

$$\text{SGZ} \cdot \alpha_z = \alpha_z \quad \text{SGZ} \cdot \alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{SGZ} \cdot \beta_z = -\beta_z \quad \text{SGZ} \cdot \beta_z = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The spin states  $\alpha_x$  and  $\beta_x$  are eigenfunctions of the SGX operator with eigenvalues +1 and -1, respectively:

$$\text{SGX} \cdot \alpha_x = \alpha_x \quad \text{SGX} \cdot \alpha_x = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \text{SGX} \cdot \beta_x = -\beta_x \quad \text{SGX} \cdot \beta_x = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$$

The spin states  $\alpha_x$  and  $\beta_x$  are not eigenfunctions of the SGZ operator as is shown below.

$$SGZ \cdot \alpha_x = \beta_x \quad SGZ \cdot \alpha_x = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad SGZ \cdot \beta_x = \alpha_x \quad SGZ \cdot \beta_x = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

And, of course, the spin states  $\alpha_z$  and  $\beta_z$  are not eigenfunctions of the SGX operator as is shown below.

$$SGX \cdot \alpha_z = \beta_z \quad SGX \cdot \alpha_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad SGX \cdot \beta_z = \alpha_z \quad SGX \cdot \beta_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

### The Results Predicted for the Interaction with the SGX Magnet

The probability that an  $\alpha_z$  Ag atom will emerge spin-up after passing through a SGX magnet:

$$\left| \langle \alpha_x | SGX | \alpha_z \rangle \right|^2 = 1/2 \quad \left( \left| \alpha_x^T \cdot SGX \cdot \alpha_z \right| \right)^2 \rightarrow 1/2$$

The probability that an  $\alpha_z$  Ag atom will emerge spin-down after passing through a SGX magnet:

$$\left| \langle \beta_x | SGX | \alpha_z \rangle \right|^2 = 1/2 \quad \left( \left| \beta_x^T \cdot SGX \cdot \alpha_z \right| \right)^2 \rightarrow 1/2$$

### The Results Predicted for the Interaction with the Final SGZ Magnet

The probability that an  $\alpha_x$  Ag atom will emerge spin-up after passing through a SGZ magnet:

$$\left| \langle \alpha_z | SGZ | \alpha_x \rangle \right|^2 = 1/2 \quad \left( \left| \alpha_z^T \cdot SGZ \cdot \alpha_x \right| \right)^2 \rightarrow 1/2$$

The probability that an  $\alpha_x$  Ag atom will emerge spin-down after passing through a SGZ magnet:

$$\left| \langle \beta_z | SGZ | \alpha_x \rangle \right|^2 = 1/2 \quad \left( \left| \beta_z^T \cdot SGZ \cdot \alpha_x \right| \right)^2 \rightarrow 1/2$$

### The Results Predicted for the Interaction with the First SGZ Magnet

Now we deal with the most difficult part of the analysis. How does quantum mechanics predict what will happen when an unpolarized spin beam encounters the initial SGZ magnet. As mention earlier, an unpolarized spin beam is a **mixture** of all possible spin polarizations. We proceed by introducing the density operator, which is a more general quantum mechanical construct than the wavefunction that can be used to represent both pure states and mixtures, as shown below.

$$\hat{\rho}_{pure} = |\Psi\rangle\langle\Psi| \quad \hat{\rho}_{mixed} = \sum p_i |\Psi_i\rangle\langle\Psi_i|$$

In the equation on the right,  $p_i$  is the fraction of the mixture in the state  $\Psi_i$ . It is not difficult to demonstrate the origin of the density operator and its utility in quantum mechanical calculations. The

expectation value for a pure state  $\Psi$  for the measurement operator  $\mathbf{A}$  is traditionally written as follows.

$$\langle A \rangle = \langle \Psi | \hat{A} | \Psi \rangle$$

Expansion of  $\Psi$  in the eigenfunctions of the measurement operator, followed by rearrangement of the brackets yields the calculation of the expectation value of  $\mathbf{A}$  in terms of the product of density operator and the measurement operator,  $\mathbf{A}$ .

$$\langle A \rangle = \langle \Psi | \hat{A} | \Psi \rangle = \sum_a \langle \Psi | \hat{A} | a \rangle \langle a | \Psi \rangle = \sum_a \langle a | \Psi \rangle \langle \Psi | \hat{A} | a \rangle = \sum_a \langle a | \hat{\rho} \hat{A} | a \rangle = \text{Trace}(\hat{\rho} \hat{A})$$

We now show that the traditional method and the method using the trace function give the same result for the z-direction spin eigenfunctions.

$$\alpha_z^T \cdot \text{SGZ} \cdot \alpha_z = 1 \qquad \text{tr}(\alpha_z \cdot \alpha_z^T \cdot \text{SGZ}) = 1$$

$$\beta_z^T \cdot \text{SGZ} \cdot \beta_z = -1 \qquad \text{tr}(\beta_z \cdot \beta_z^T \cdot \text{SGZ}) = -1$$

An unpolarized beam can be written as a 50-50 mixture of any of the orthogonal spin eigenfunctions -  $\alpha_z$  and  $\beta_z$ , or  $\alpha_x$  and  $\beta_x$ , or  $\alpha_y$  and  $\beta_y$ . According to the previous definition, the density operator for an unpolarized spin beam is as follows.

$$\hat{\rho}_{mix} = \frac{1}{2} |\alpha_z\rangle \langle \alpha_z| + \frac{1}{2} |\beta_z\rangle \langle \beta_z| = \frac{1}{2} |\alpha_x\rangle \langle \alpha_x| + \frac{1}{2} |\beta_x\rangle \langle \beta_x| = \frac{1}{2} |\alpha_y\rangle \langle \alpha_y| + \frac{1}{2} |\beta_y\rangle \langle \beta_y|$$

Fifty percent of the silver atoms are deflected toward the north pole ( $\alpha_z$ , eigenvalue +1) and fifty percent toward the south pole ( $\beta_z$ , eigenvalue -1). Therefore, the expectation value should be zero as is calculated below using both z- and x- spin directions.

$$\text{tr}\left[\left(\frac{1}{2} \cdot \alpha_z \cdot \alpha_z^T + \frac{1}{2} \cdot \beta_z \cdot \beta_z^T\right) \cdot \text{SGZ}\right] = 0 \qquad \text{tr}\left[\left(\frac{1}{2} \cdot \alpha_x \cdot \alpha_x^T + \frac{1}{2} \cdot \beta_x \cdot \beta_x^T\right) \cdot \text{SGZ}\right] = 0$$

An equivalent method of obtaining the same result is shown below.

$$\frac{1}{2} \cdot \alpha_z^T \cdot \text{SGZ} \cdot \alpha_z + \frac{1}{2} \cdot \beta_z^T \cdot \text{SGZ} \cdot \beta_z = 0 \qquad \frac{1}{2} \cdot \alpha_z^T \cdot \text{SGZ} \cdot \alpha_z + \frac{1}{2} \cdot \beta_z^T \cdot \text{SGZ} \cdot \beta_z = 0$$

One of the attractive features of matrix mechanics at the level of electron spin states or photon polarization states is that all of the necessary calculations can easily be performed without the aid of a computer. The reader is encouraged to confirm this assertion by repeating all the calculations in this tutorial by hand.