The purpose of this tutorial is to analyze the Stern-Gerlach experiment using matrix mechanics. The figure below is taken (and modified) from Thomas Engel's text, *Quantum Chemistry and Spectroscopy*.

Silver atoms are deflected by an inhomogeneous magnetic field because of the two-valued magnetic moment associated with their unpaired 5s electron ([Kr]5s\(^1\)4d\(^{10}\)). The beam of silver atoms entering the Stern-Gerlach magnet oriented in the z-direction (SGZ) on the left is unpolarized. This means it is a mixture of Ag atoms with random spin orientations. As such, it is impossible to write a quantum mechanical wavefunction for this initial state. Consequently, equation 6.2 in our text is not a valid representation of the initial state of a silver atom.

This situation is exactly analogous to the three-polarizer demonstration described in a previous tutorial (http://www.users.csbsju.edu/~frioux/q-intro/polar-append.pdf). Light emerging from an incandescent light bulb is unpolarized, a mixture of all possible polarization angles, so we can't write a wave function for it. The first Stern-Gerlach magnet plays the same role as the first polarizer, it forces the Ag atoms into one of measurement eigenstates - spin-up or spin-down in the z-direction. The only difference is that in the three-polarizer demonstration only one state was created - vertical polarization. Both demonstrations illustrate an important quantum mechanical postulate - the only values that are observed in a measurement are the eigenvalues of the measurement operator.

To continue with the analysis of the Stern-Gerlach demonstration we need vectors to represent the various spin states of the Ag atoms. We will restrict our attention to the x- and z- spin directions, although the spin states for the y-direction are also available.

### Spin Eigenfunctions

**Spin-up in the z-direction:** \( \alpha_z := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)  
**Spin-down in the z-direction:** \( \beta_z := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

**Spin-up in the x-direction:** \( \alpha_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)  
**Spin-down in the x-direction:** \( \beta_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

After the initial SGZ magnet, the spin-up beam \( (\alpha_z) \) enters a magnet oriented in the x-direction, SGX. The \( \alpha_z \) beam splits into \( \alpha_x \) and \( \beta_x \) beams of equal intensity, because \( \alpha_z \) is a superposition of the x-direction spin eigenstates as shown below.

\[
\alpha_z = \frac{1}{\sqrt{2}} (\alpha_x + \beta_x)  
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]
\]
Next the $\alpha_x$ beam is directed toward a second SGZ magnet and splits into two equal $\alpha_z$ and $\beta_z$ beams, because $\alpha_x$ is a superposition of the $\alpha_z$ and $\beta_z$ spin states.

$$\alpha_x = \frac{1}{\sqrt{2}} (\alpha_z + \beta_z) \quad \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

In this approach we have analyzed the inputs to the Stern-Gerlach magnets. In the operator approach that follows we analyze the outputs of the SG magnets.

**Operator Approach**

We can also use operators (matrices) to analyze this experiment. The matrix operators associated with the two Stern-Gerlach magnets are shown below.

**SGZ operator:**

$$\text{SGZ} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**SGX operator:**

$$\text{SGX} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The SGX magnet operates on the $\alpha_z$ state, changing it to $\beta_z$ which is a superposition of $\alpha_x$ and $\beta_x$. The minus sign appears in the superposition because the eigenvalue of $\beta_x$ is -1.

$$\text{SGX} \cdot \alpha_z = \beta_z = \frac{1}{\sqrt{2}} (\alpha_x - \beta_x)$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

The final SGZ magnet operates on the $\alpha_x$ state, changing it to $\beta_x$ which is a superposition of $\alpha_z$ and $\beta_z$. The minus sign appears in the superposition because the eigenvalue of $\beta_z$ is -1.

$$\text{SGZ} \cdot \alpha_x = \beta_x = \frac{1}{\sqrt{2}} (\alpha_z - \beta_z)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

From a classical perspective, after the SGX magnet it might be assumed that the Ag atoms are in the spin state $|\alpha_z\rangle |\alpha_x\rangle$, in other words after the SGZ and SGX magnets the Ag 5s electrons have well-defined values for spin in both the z- and x-directions. However, the SGZ and SGX operators do not commute, meaning that they cannot have simultaneous eigenstates.

$$\text{SGX} \cdot \text{SGZ} - \text{SGZ} \cdot \text{SGX} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

The spin state entering the second SGZ magnet is simply $\alpha_x$, an eigenstate of the SGX operator, not simultaneously an eigenstate of SGZ and SGX.