Quantum Teleportation Examined with Tensor Algebra

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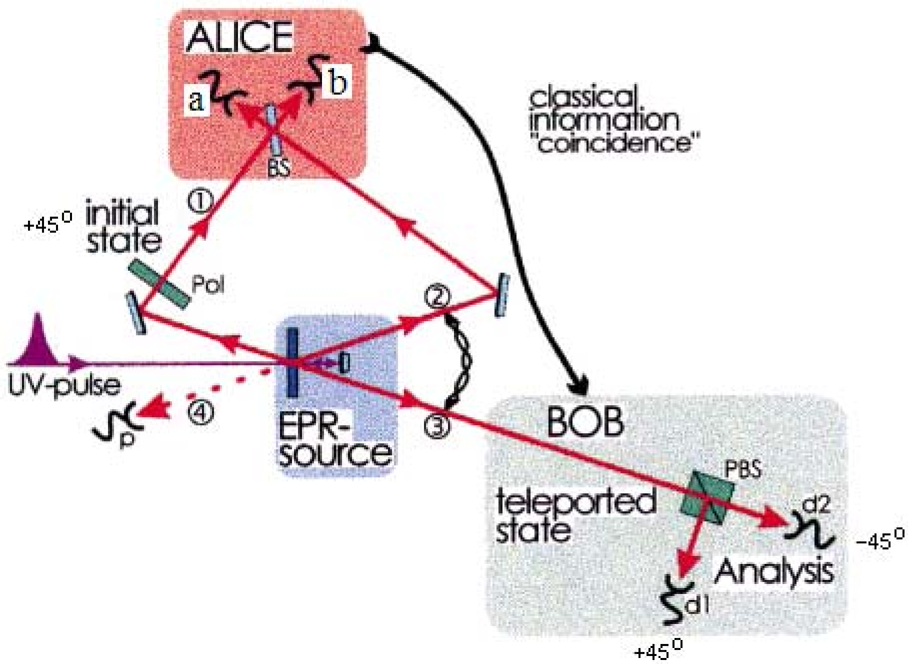
Alice wishes to teleport photon 1 in the polarization state $\alpha|h\rangle_1 + \beta|v\rangle_1$ to Bob. The letters *h* and *v* refer to the horizontal and vertical polarization states and α and β are complex coefficients such that the sum of the squares of their absolute magnitudes is unity. In the initial teleportation step, Alice and Bob prepare an entangled state involving photons 2 and 3, $\frac{1}{\sqrt{2}}\left[|h\rangle_2|v\rangle_3 - |v\rangle_2|h\rangle_3\right]$, as shown in the figure below [Nature **390**, 576 (1997)].



Alice has photon 2 and Bob photon 3, but because they are in an entangled state the photons do not have well-defined individual polarization states. Alice arranges for photons 1 and 2 to arrive simultaneously at opposite sides of a beam splitter. This gives rise to the following state.

$$\left(\alpha|h\rangle_1 + \beta|v\rangle_1\right)\left(\frac{|h\rangle_2|v\rangle_3 - |v\rangle_2|h\rangle_3}{\sqrt{2}}\right)$$

In the tensor algebra analysis used here the following vector definitions will be used for the horizontal and vertical polarization states.

$$|h\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |v\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Expressing the three-photon state in tensor format yields (see Appendix for mathematical details),

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\otimes\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\otimes\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\otimes\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\\alpha\\-\alpha\\0\\0\\\beta\\-\beta\\0\end{pmatrix}$$

Alice now makes a Bell-state measurement on photons 1 and 2. Bell states are the four maximally entangled h-v polarization states of photons 1 and 2. In tensor format they are as follows.

$$\left|\Phi^+\right\rangle=\frac{1}{\sqrt{2}}\left[|h\rangle_1|h\rangle_2+|v\rangle_1|v\rangle_2\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\left|\Phi^-\right\rangle=\frac{1}{\sqrt{2}}\left[|h\rangle_1|h\rangle_2-|v\rangle_1|v\rangle_2\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\left|\Psi^+\right\rangle=\frac{1}{\sqrt{2}}\left[|h\rangle_1|v\rangle_2+|v\rangle_1|h\rangle_2\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\left|\Psi^-\right\rangle=\frac{1}{\sqrt{2}}\left[|h\rangle_1|v\rangle_2-|v\rangle_1|h\rangle_2\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

The three-photon state is now written as the superposition of the Bell states and Bob's photon as shown below. Photon 3 carries the teleported state that results from Alice's Bell state measurement. We see that there are four equally likely experimental outcomes.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \alpha\\ -\alpha\\ 0\\ 0\\ \beta\\ -\beta\\ 0\end{pmatrix}=\left|\Phi^+\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}+\left|\Phi^-\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}+\left|\Psi^+\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}+\left|\Psi^-\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}$$

Each term on the right side is now written in tensor format so that its projection onto the three-photon state can be calculated.

$$\left|\Phi^+\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}h_3\\ v_3\\ 0\\ 0\\ 0\\ 0\\ h_3\\ v_3\end{pmatrix}\qquad \left|\Phi^-\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ -1\end{pmatrix}\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}h_3\\ v_3\\ 0\\ 0\\ 0\\ 0\\ -h_3\\ -v_3\end{pmatrix}$$

$$\left|\Psi^+\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ 1\\ 0\end{pmatrix}\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ h_3\\ v_3\\ h_3\\ v_3\\ 0\\ 0\end{pmatrix}\qquad \left|\Psi^-\right\rangle\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ -1\\ 0\end{pmatrix}\otimes\begin{pmatrix}h_3\\ v_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ h_3\\ v_3\\ -h_3\\ -v_3\\ 0\\ 0\end{pmatrix}$$

The projections are calculated as follows. We see that the probability amplitudes are ½, and therefore the probabilities of occurrence are each ¼.

$$\frac{1}{2}\begin{pmatrix}0 & \alpha & -\alpha & 0 & 0 & \beta & -\beta & 0\end{pmatrix}\begin{pmatrix}h_3\\ v_3\\ 0\\ 0\\ 0\\ 0\\ h_3\\ v_3\end{pmatrix}=\frac{1}{2}\left(-\beta h_3+\alpha v_3\right)\qquad \frac{1}{2}\begin{pmatrix}0 & \alpha & -\alpha & 0 & 0 & \beta & -\beta & 0\end{pmatrix}\begin{pmatrix}h_3\\ v_3\\ 0\\ 0\\ 0\\ 0\\ -h_3\\ -v_3\end{pmatrix}=\frac{1}{2}\left(\beta h_3+\alpha v_3\right)$$

$$\begin{pmatrix}0 & \alpha & -\alpha & 0 & 0 & \beta & -\beta & 0\end{pmatrix}\begin{pmatrix}0\\0\\h_3\\v_3\\h_3\\v_3\\0\\0\end{pmatrix}=\frac{1}{2}(-\alpha h_3+\beta v_3) \qquad \frac{1}{2}\begin{pmatrix}0 & \alpha & -\alpha & 0 & 0 & \beta & -\beta & 0\end{pmatrix}\begin{pmatrix}0\\0\\h_3\\v_3\\-h_3\\-v_3\\0\\0\end{pmatrix}=-\frac{1}{2}(\alpha h_3+\beta v_3)$$

The consequences of Alice's Bell state measurement are summarized as follows.

$$\left|\Phi^+\right\rangle \to \frac{1}{2}\begin{pmatrix}-\beta\\ \alpha\end{pmatrix} \qquad \left|\Phi^-\right\rangle \to \frac{1}{2}\begin{pmatrix}\beta\\ \alpha\end{pmatrix} \qquad \left|\Psi^+\right\rangle \to \frac{1}{2}\begin{pmatrix}-\alpha\\ \beta\end{pmatrix} \qquad \left|\Psi^-\right\rangle \to -\frac{1}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

We will restrict our attention to the last case which, of course, occurs 25% of the time. After Alice measures $\left|\Psi^-\right\rangle$ she communicates this to Bob by classical means and he knows that he has received photon 1's polarization state and no further action is required by him.

But how does Alice know she has measured $\left|\Psi^-\right\rangle$? Of the four Bell states, it is the only one that is antisymmetric with respect to the interchange of the photon labels. Thus, in spite of the fact that photons are individually bosons, this entangled photon state is fermionic. Collectively the photons in this Bell state are behaving as fermions. This means that they can't be in the same measurement state at the same time. If photon 1 is detected at **a**, then photon 2 will be detected at **b**. Therefore, if Alice observes **a-b** coincidences it means photons 1 and 2 are in the antisymmetric Bell state and that photon 1's polarization state has been teleported to Bob's photon (#3).

Alice's other three Bell state measurement outcomes are more difficult to analyze and require further action on Bob's part in order to receive photon 1's polarization state. They will not be considered here.

Appendix

The tensor product of three qubits is carried out as follows.

$$\begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}c\\d\end{pmatrix}\otimes\begin{pmatrix}e\\f\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}ce\\cf\\de\\df\end{pmatrix}=\begin{pmatrix}ace\\acf\\ade\\adf\\bce\\bcf\\bde\\bdf\end{pmatrix}$$