

Visualizing the Difference Between a Superposition and a Mixture

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The superposition principle, as Feynman said, is at the heart of quantum mechanics. While its mathematical expression is simple, its true meaning is difficult to grasp. For example, given a linear superposition (not normalized) of two states,

$$|\Psi\rangle = |\phi_1\rangle + |\phi_2\rangle$$

one might assume that it represents a mixture of ϕ_1 and ϕ_2 . In other words, half of the quons [1] are in state ϕ_1 and half in ϕ_2 . However, the correct quantum mechanical interpretation of this equation is that the system represented by Ψ is simultaneously in the states ϕ_1 and ϕ_2 , properly weighted.

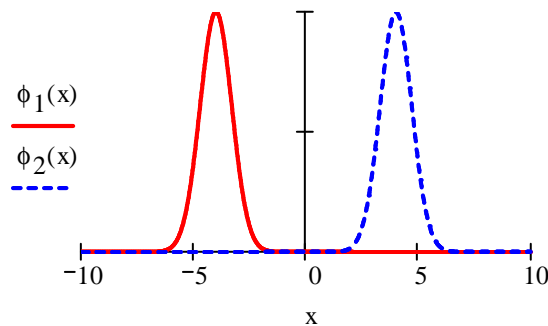
A mixture, half ϕ_1 and half ϕ_2 , or any other ratio, cannot be represented by a wavefunction. It requires a density operator, which is a more general quantum mechanical construct that can be used to represent both pure states (superpositions) and mixtures, as shown below.

$$\hat{\rho}_{pure} = |\Psi\rangle\langle\Psi| \quad \hat{\rho}_{mixed} = \sum p_i |\Psi_i\rangle\langle\Psi_i|$$

In the equation on the right, p_i is the fraction of the mixture in the state Ψ_i .

To illustrate how these equations distinguish between a mixture and a superposition, we will consider a superposition and a mixture of equally weighted gaussian functions representing one-dimensional wave packets. The normalization constants are omitted in the interest of mathematical clarity. The gaussians are centered at $x = \pm 4$.

$$\phi_1(x) := \exp[-(x+4)^2] \quad \phi_2(x) := \exp[-(x-4)^2]$$



To visualize how the density operator discriminates between a superposition and a mixture, we calculate its matrix elements in coordinate space for the 50-50 superposition and mixture of ϕ_1 and ϕ_2 . The superposition is considered first.

$$\Psi(x) := \phi_1(x) + \phi_2(x)$$

The matrix elements of this pure state are calculated as follows.

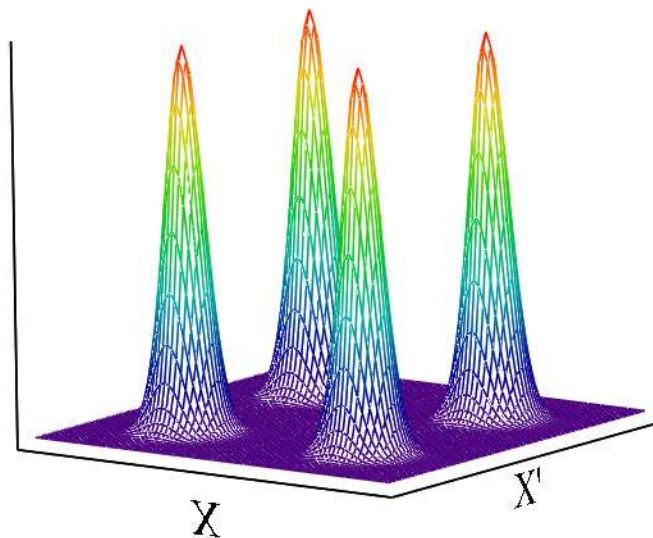
$$\rho_{pure} = \langle x | \hat{\rho}_{pure} | x' \rangle = \langle x | \Psi \rangle \langle \Psi | x' \rangle$$

Looking at the right side we see that the matrix elements are the product of the probability amplitudes of a quon in state Ψ being at x and x' . Next we display the density matrix graphically.

$$\text{DensityMatrixPure}(x, x') := \Psi(x) \cdot \Psi(x')$$

$$x_0 := 8 \quad N := 80 \quad i := 0..N \quad x_i := -x_0 + \frac{2 \cdot x_0 \cdot i}{N} \quad j := 0..N \quad x'_j := -x_0 + \frac{2 \cdot x_0 \cdot j}{N}$$

$$\text{DensityMatrixPure}_{i,j} := \text{DensityMatrixPure}(x_i, x'_j)$$



DensityMatrixPure

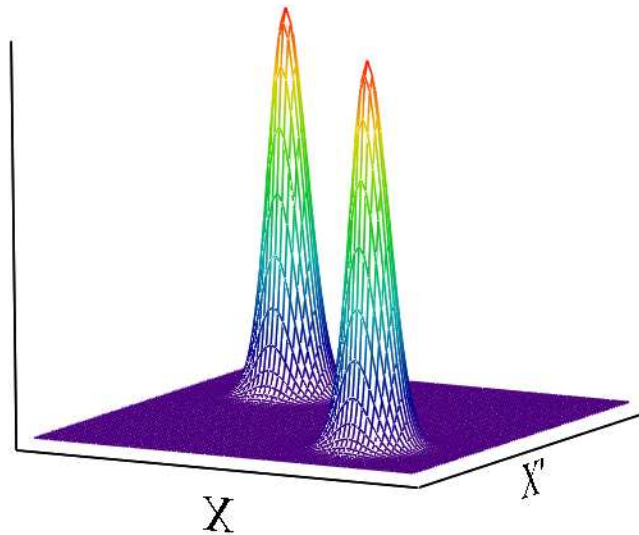
The presence of off-diagonal elements in this density matrix is the signature of a quantum mechanical superposition. For example, from the quantum mechanical perspective bi-location is possible.

Now we turn our attention to the density matrix of a mixture of gaussian states.

$$\rho_{mix} = \langle x | \hat{\rho}_{mix} | x' \rangle = \sum_i p_i \langle x | \phi_i \rangle \langle \phi_i | x' \rangle = \frac{1}{2} \langle x | \phi_1 \rangle \langle \phi_1 | x' \rangle + \frac{1}{2} \langle x | \phi_2 \rangle \langle \phi_2 | x' \rangle$$

$$\text{DensityMatrixMix}(x, x') := \frac{\phi_1(x) \cdot \phi_1(x') + \phi_2(x) \cdot \phi_2(x')}{2}$$

$$\text{DensityMatrixMix}_{i,j} := \text{DensityMatrixMix}(x_i, x'_j)$$



DensityMatrixMix

The obvious difference between a superposition and a mixture is the absence of off-diagonal elements in the mixed state. This indicates the mixture is in a definite but unknown state; it is an example of classical ignorance.

An equivalent way to describe the difference between a superposition and a mixture, is to say that to calculate the probability of measurement outcomes for a superposition you add the probability amplitudes and square the sum. For a mixture you square the individual probability amplitudes and sum the squares.

1. Nick Herbert (*Quantum Reality*, page 64) suggested "quon" be used to stand for a generic quantum object. "A quon is any entity, no matter how immense, that exhibits both wave and particle aspects in the peculiar quantum manner.