The Bohr Model of the Earth-Sun System*

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Assuming the earth executes a circular orbit of radius $r$ about the sun and has a deBroglie wavelength given by $\hbar/mv$, yields a quantum mechanical kinetic energy for the earth which is the first term in the total energy expression below. The potential energy of the earth-sun interaction is well-known and is the second term in the total energy expression.

$$E = \frac{n^2 \cdot \hbar^2}{8 \cdot \pi^2 \cdot Me \cdot r^2} - \frac{G \cdot Me \cdot Ms}{r}$$

Setting the first derivative of the energy with respect to $r$ equal to zero, yields the allowed values of $r$ in terms of the quantum number, $n$.

$$\frac{d}{dr} \left( \frac{n^2 \cdot \hbar^2}{8 \cdot \pi^2 \cdot Me \cdot r^2} - \frac{G \cdot Me \cdot Ms}{r} \right) = 0 \quad \text{solve } r \rightarrow \frac{1}{4} \cdot n^2 \cdot \frac{\hbar^2}{G \cdot Me^2 \cdot Ms \cdot \pi^2}$$

Substitution of this value of $r$ in the total energy expression yields the energy of the earth-sun system as a function of the quantum number, $n$, Planck's constant, the gravitational constant, and the masses of the earth and the sun.

$$E = \frac{n^2 \cdot \hbar^2}{8 \cdot \pi^2 \cdot Me \cdot r^2} - \frac{G \cdot Me \cdot Ms}{r} \quad \text{substitute } r = \frac{1}{4} \cdot n^2 \cdot \frac{\hbar^2}{G \cdot Me^2 \cdot Ms \cdot \pi^2} \rightarrow E = \frac{-2}{n^2} \cdot \frac{\hbar^2}{G \cdot Me^3 \cdot Ms^2}$$

Data for the earth-sun system assuming a circular earth orbit:

- Mass of the earth: $Me := 5.974 \cdot 10^{24} \cdot \text{kg}$
- Mass of the sun: $Ms := 1.989 \cdot 10^{30} \cdot \text{kg}$
- Earth orbit radius: $r := 1.496 \cdot 10^{11} \cdot \text{m}$
- Gravitational constant: $G := 6.674 \cdot 10^{-11} \cdot \text{N} \cdot \text{m}^2/\text{kg}^2$
- Planck's constant: $\hbar := 6.62608 \cdot 10^{-34} \cdot \text{J} \cdot \text{s}$

Given the radius of the earth's orbit listed above, calculate the earth's quantum number.

$$r = \frac{1}{4} \cdot n^2 \cdot \frac{\hbar^2}{G \cdot Me^2 \cdot Ms \cdot \pi^2} \quad \text{solve } n \rightarrow \frac{1}{2} \cdot m \cdot \frac{1}{s \cdot \text{J}} \rightarrow \frac{1}{2} \cdot m \cdot \frac{1}{s \cdot \text{J}} \left[ 2.52e75 \cdot (\text{m} \cdot \text{kg} \cdot \text{N}) \right] = \left( 2.52 \times 10^{74} \right) \left( -2.52 \times 10^{74} \right)$$

The positive root for $n$ is used to calculate the energy of the earth-sun system.

$$n := 2.52 \cdot 10^{74} \quad E := \frac{n^2 \cdot \hbar^2}{8 \cdot \pi^2 \cdot Me \cdot r^2} - \frac{G \cdot Me \cdot Ms}{r} \quad E = -2.66 \times 10^{33} \text{ J}$$
According to the virial theorem the classical expression for the energy of the earth-sun system with earth orbit radius $r$ is half the potential energy. Note that this gives a value which is in agreement with the Bohr model for the earth-sun system. Is this a legitimate example of the correspondence principle?

$$E := -\frac{G M_e M_s}{2r} \quad \quad E = -2.65 \times 10^{33} \text{ J}$$

*Johnson and Pedersen, Problems and Solutions in Quantum Chemistry and Physics, pages 26-27.*