A Quantum Mechanical Interpretation of Single-slit Diffraction

Or, Using Diffration Phenomena to Illustrate the Uncertainty Principle

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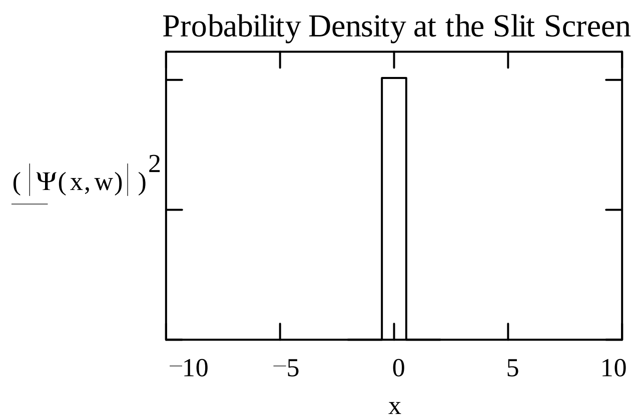
Diffraction has a simple quantum mechanical interpretation based on the uncertianty principle. Or we could say diffraction is an excellent way to illustrate the uncertainty principle.

A screen with a single slit of width, w, is illuminated with a coherent photon or particle beam. The normalized coordinate-space wave function at the slit screen is,

$$\Psi(x,w) := \text{if}\left[\left(x \geq -\frac{w}{2}\right)\cdot\left(x \leq \frac{w}{2}\right), \frac{1}{\sqrt{w}}, 0\right]$$

The coordinate-space probability density, $|\Psi(x,w)|^2$, is displayed for a slit of unit width below

$$w := 1 \qquad x := -2\cdot w, -2\cdot w + .01 .. 2\cdot w$$



Since the slit-screen measures position, it localizes the incident beam in the x-direction. Accoring to the uncertainty principle, because position and momentum are incompatible, or conjugate, observables, this measurement must be accompanied by a delocalization of the x-component of the momentum. To see this it only necessary to Fourier transform Ψ(x,w) into momentum space to obtain the momentum wave function, $\Phi(p_x,w)$.

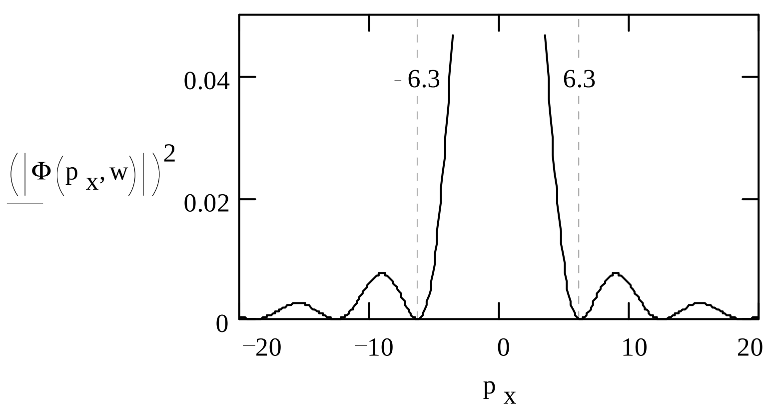
$$\Phi(p_x, w) := \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_{-\frac{w}{2}}^{\frac{w}{2}} \exp(-i\cdot p_x\cdot x)\cdot\Psi(x,w)\,dx$$

Evaluation of this integral yields:

$$\Phi\left(p_x, w\right) := \sqrt{\frac{2}{\pi \cdot w}} \cdot \frac{\sin\left(\frac{p_x \cdot w}{2}\right)}{p_x}$$

It is the momentum distribution, $|\Phi(p_x,w)|^2$, shown below that is projected onto the detection screen. Thus, a position measurement at the detection screen is also effectively a measure of the x-component of the particle momentum.

$$p_x := -20, -19.9 .. 20$$

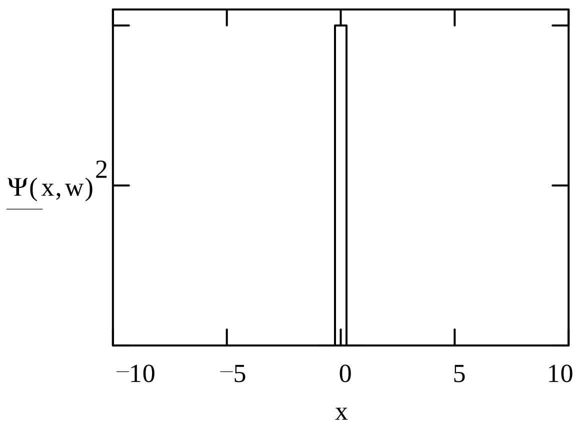


Momentum Distribution at the Slit Screen

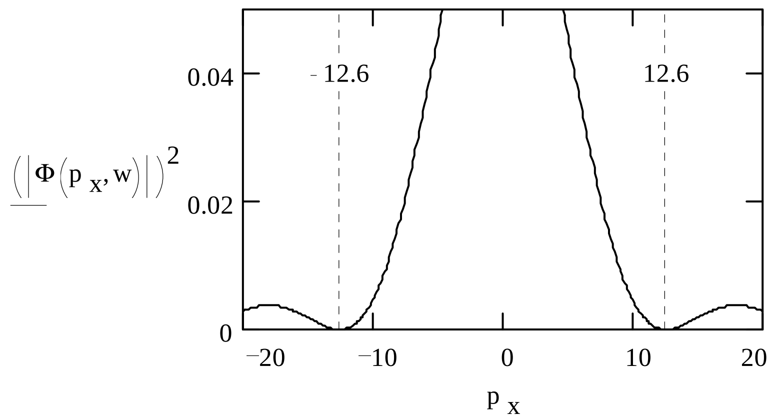
In this figure we see the spread in momentum required by the uncertainty principle, plus interference fringes due to the fact that the incident beam can imerge from any where within the slit, allowing for constructive and destructive interference. If the slit width is decreased the position is more precisely known and the uncertainty principle demands a broadening in the momentum distribution as shown below.

Equating uncertainty in position with slit width and uncertainty in momentum with the width of the intense center of the diffraction pattern, we have in atomic units: $\Delta x \Delta p_x = 12.6$. If the slit width is decreased the position is more precisely known and the uncertainty principle demands a broadening in the momentum distribution as shown below. Again we find the product of the uncertainties is 12.6.

Slit width = 0.5 $\quad w := .5 \quad x := -2\cdot w, -2\cdot w + .01 .. 2\cdot w$



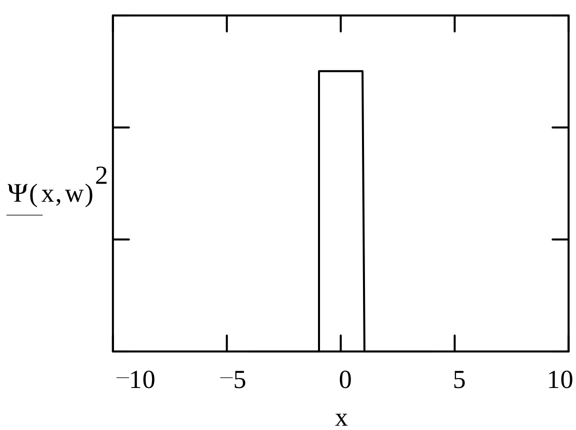
Probability Density at the Slit Screen



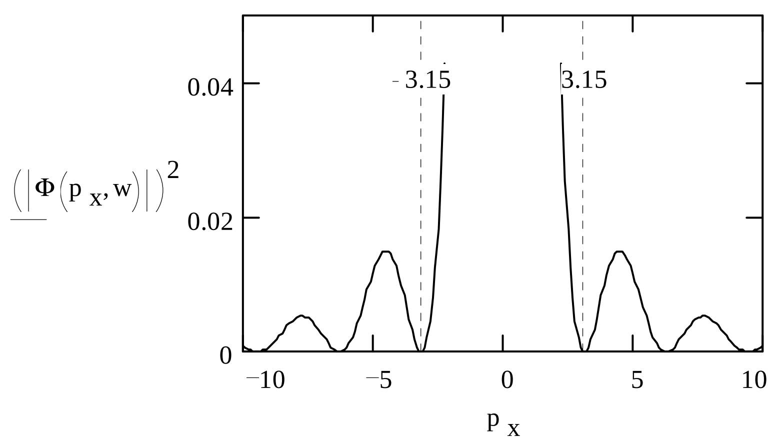
Momentum Distribution at the Slit Screen

Naturally if the slit width is increased the position uncertainty increases and the uncertainty in momentum decrease yielding again $\Delta x \Delta p_x = 12.6$.

Slit width = 2 $\quad w := 2 \quad x := -2\cdot w, -2\cdot w + .01 .. 2\cdot w$



Probability Density at the Slit Screen



Momentum Distribution at the Slit Screen

The method used here to calculate single-slit diffraction patterns (momentum-space distribution functions) is easily extended to multiple slits, and also to diffraction at two-dimensional masks with a variety of hole geometries.

Primary source: "Quantum interference with slits" by Thomas Marcella which appeared in *European Journal of Physics* **23**, 615-621 (2002).

See also: "Calculating diffraction patterns," by Frank Rioux in *European Journal of Physics*, **24**, N1-N3 (2003).

"Experimental verification of the Heisenberg uncertainty Principle for hot fullerene molecules", O. Nairz, M. Arndt, and A. Zeilinger, *Phys. Rev. A*, **65**, 032109 (2002).