

The Wigner Function for the One-dimensional Hydrogen Atom

Frank Rioux

The energy operator for the one-dimensional hydrogen atom in atomic units is: $\frac{-1}{2} \cdot \frac{d^2}{dx^2} - \frac{1}{x}$

The ground eigenstate is: $\Psi(x) := 2 \cdot x \cdot \exp(-x)$

The ground state energy is $-0.5 E_H$. $\frac{-1}{2} \cdot \frac{d^2}{dx^2} \Psi(x) - \frac{1}{x} \cdot \Psi(x) = E \cdot \Psi(x)$ solve, $E \rightarrow \frac{-1}{2}$

The momentum wave function is generated by the following Fourier transform of the coordinate space wave function.

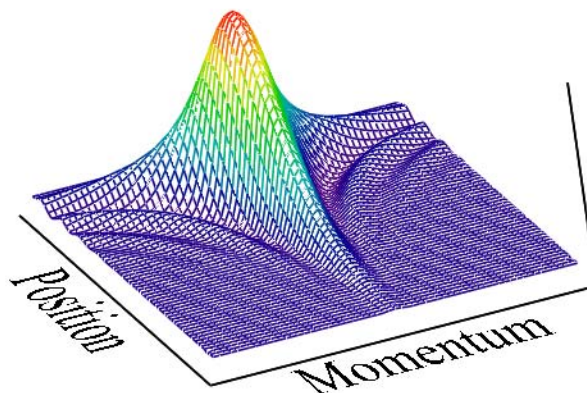
$$\Phi(p) := \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int_0^{\infty} \exp(-i \cdot p \cdot x) \cdot \Psi(x) dx \rightarrow \frac{\frac{1}{2^2}}{\pi^2 \cdot (i \cdot p + 1)^2}$$

The Wigner function for the hydrogen atom ground state is generated using the momentum wave function.

$$W(x, p) := \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} \overline{\Phi\left(p + \frac{s}{2}\right)} \cdot \exp(-i \cdot s \cdot x) \cdot \Phi\left(p - \frac{s}{2}\right) ds$$

The Wigner distribution is displayed graphically.

$$N := 60 \quad i := 0..N \quad x_i := \frac{6 \cdot i}{N} \quad j := 0..N \quad p_j := -5 + \frac{10 \cdot j}{N} \quad \text{Wigner}_{i,j} := W(x_i, p_j)$$



Wigner

One of the interesting features of doing quantum mechanics with the Wigner distribution is that the position and momentum operators retain their classical forms; they are both multiplicative operators. By comparison in the coordinate representation position is multiplicative and momentum is differential. In the momentum representation it's the reverse. This is illustrated below with the following calculations.

Phase space calculations using the Wigner distribution:

The Wigner distribution is normalized:
$$\int_{-\infty}^{\infty} \int_0^{\infty} W(x, p) dx dp = 1$$

The expectation value for position:
$$\int_{-\infty}^{\infty} \int_0^{\infty} x \cdot W(x, p) dx dp = 1.5$$

The expectation value for momentum:
$$\int_{-\infty}^{\infty} \int_0^{\infty} p \cdot W(x, p) dx dp = 0$$

The expectation value for kinetic energy:
$$\int_{-\infty}^{\infty} \int_0^{\infty} \frac{p^2}{2} \cdot W(x, p) dx dp = 0.5$$

The expectation value for potential energy:
$$\int_{-\infty}^{\infty} \int_0^{\infty} \frac{-1}{x} \cdot W(x, p) dx dp = -1$$

The same calculations made with the coordinate space wave function:

The coordinate wave function is normalized:
$$\int_0^{\infty} \Psi(x)^2 dx = 1$$

The expectation value for position:
$$\int_0^{\infty} x \cdot \Psi(x)^2 dx = 1.5$$

The expectation value for momentum:
$$\int_0^{\infty} \Psi(x) \cdot \frac{1}{i} \cdot \frac{d}{dx} \Psi(x) dx = 0$$

The expectation value for kinetic energy:
$$\int_0^{\infty} \Psi(x) \cdot \frac{-1}{2} \cdot \frac{d^2}{dx^2} \Psi(x) dx = 0.5$$

The expectation value for potential energy:

$$\int_0^{\infty} \frac{-1}{x} \cdot \Psi(x)^2 dx = -1$$

The same calculations made with the momentum space wave function:

The momentum wave function is normalized:

$$\int_{-\infty}^{\infty} (|\Phi(p)|)^2 dp = 1$$

The expectation value for position:

$$\int_{-\infty}^{\infty} \overline{\Phi(p)} \cdot i \cdot \frac{d}{dp} \Phi(p) dp = 1.5$$

The expectation value for momentum:

$$\int_{-\infty}^{\infty} p \cdot (|\Phi(p)|)^2 dp = 0$$

The expectation value for kinetic energy:

$$\int_{-\infty}^{\infty} \frac{p^2}{2} \cdot (|\Phi(p)|)^2 dp = 0.5$$

The expectation value for potential energy:

I don't know how to make this calculation.