Wigner Distribution for the Particle in a Box

The Wigner function is a quantum mechanical phase-space quasi-probability function. It is called a quasi-probability function because it can take on negative values, which have no classical meaning in terms of probability.

The PIB eigenstates for a box of unit dimension are given by $\Psi(x,n) := \sqrt{2} \cdot \sin(n \pi \cdot x)$

For these eigenstates the Wigner distribution function is:

$$W(x,p,n) := \frac{1}{\pi} \int_{-x}^{x} \sqrt{2} \cdot \sin[n \pi \cdot (x + s)] \cdot \exp(2i \cdot s \cdot p) \cdot \sqrt{2} \cdot \sin[n \pi \cdot (x - s)] \, ds$$

Integration with respect to $s$ yields the following function:

$$W(x,p,n) := \frac{2}{\pi} \left[ \frac{\sin[2 \cdot (p - n \pi) \cdot x]}{4 (p - n \pi)} + \frac{\sin[2 \cdot (p + n \pi) \cdot x]}{4 (p + n \pi)} - \cos(2n \pi \cdot x) \cdot \frac{\sin(2p \cdot x)}{2p} \right]$$

The Wigner distribution for the $n^{th}$ eigenstate is calculated below:

$$n := 10$$

$$N := 115 \quad i := 0..N \quad x_i := \frac{i}{N} \quad j := 0..N \quad p_j := -40 + \frac{80 \cdot j}{N}$$

$$Wigner_{i,j} := \text{if}[x_i \leq .5, W(x_i,p_j,n), W((1 - x_i),p_j,n)]$$
Integration of the Wigner function over the spatial coordinate yields the momentum distribution function as is shown below.

\[
\rho(p) := \int_{0}^{1} W(x, p, n) \, dx \quad p := -40, -39.5 \ldots 40
\]

Integration of the Wigner function over the momentum coordinate yields the spatial distribution function as is shown below.

\[
\rho(x) := \int_{-51}^{50} W(x, p, n) \, dp \quad x := 0, .01 \ldots 1
\]
The Wigner distribution can be used to calculate the expectation values for position, momentum and kinetic energy.

\[
x_{\text{bar}} = \int_{-\infty}^{\infty} \int_{0}^{1} W(x, p, 1) \cdot x \, dx \, dp \quad \text{simplify} \quad x_{\text{bar}} = \frac{1}{2}
\]

\[
p_{\text{bar}} = \int_{-\infty}^{\infty} \int_{0}^{1} W(x, p, 1) \cdot p \, dx \, dp \quad \text{simplify} \quad p_{\text{bar}} = 0
\]

\[
T_{\text{bar}} = \int_{-\infty}^{\infty} \int_{0}^{1} W(x, p, 1) \cdot \frac{p^2}{2} \, dx \, dp \quad \text{simplify} \quad T_{\text{bar}} = \frac{1}{2} \cdot \pi^2
\]

**References:**
