

Examining the Wigner Distribution Using Dirac Notation

Frank Rioux

Department of Chemistry

St. John's University and College of St. Benedict

St. Joseph, MN 56374

Abstract: Expressing the Wigner distribution function in Dirac notation reveals its resemblance to a classical trajectory in phase space.

References to the Wigner distribution function [1-3] and the phase-space formulation of quantum mechanics are becoming more frequent in the pedagogical and review literature [4-26]. There have also been several important applications reported in the recent research literature [27, 28]. Other applications of the Wigner distribution are cited in Ref. 25.

The purpose of this note is to demonstrate that when expressed in Dirac notation the Wigner distribution resembles a classical phase-space trajectory. The Wigner distribution can be generated from either the coordinate- or momentum-space wave function. The coordinate-space wave function will be employed here and the Wigner transform using it is given in equation (1) for a one-dimensional example in atomic units.

$$W(p, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^* \left(x + \frac{s}{2}\right) \Psi \left(x - \frac{s}{2}\right) e^{ips} ds \quad (1)$$

In Dirac notation the first two terms of the integrand are written as follows,

$$\Psi^* \left(x + \frac{s}{2}\right) = \left\langle \Psi \left| x + \frac{s}{2} \right. \right\rangle \quad \Psi \left(x - \frac{s}{2}\right) = \left\langle x - \frac{s}{2} \left| \Psi \right. \right\rangle \quad (2)$$

Assigning $1/2\pi$ to the third term and utilizing the momentum eigenfunction in coordinate space and its complex conjugate we have,

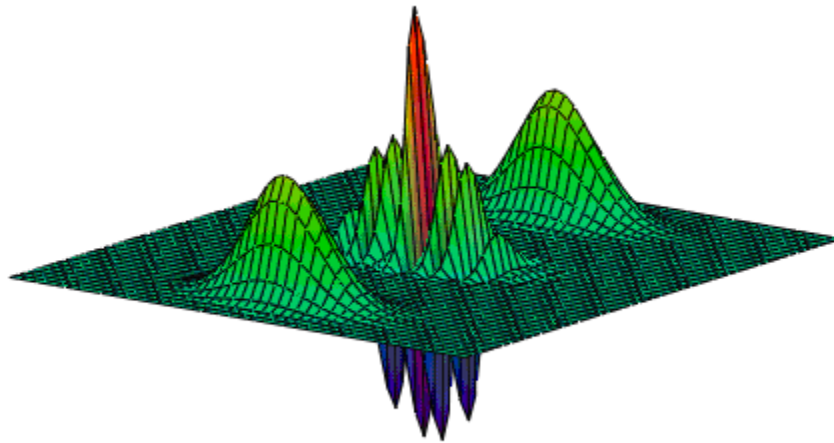
$$\frac{1}{2\pi} e^{ips} = \frac{1}{\sqrt{2\pi}} e^{ip(x+\frac{s}{2})} \frac{1}{\sqrt{2\pi}} e^{-ip(x-\frac{s}{2})} = \left\langle x + \frac{s}{2} \left| p \right. \right\rangle \left\langle p \left| x - \frac{s}{2} \right. \right\rangle \quad (3)$$

Substituting equations (2) and (3) into equation (1) yields after arrangement,

$$W(x, p) = \int_{-\infty}^{+\infty} \langle \Psi | x + \frac{s}{2} \rangle \langle x + \frac{s}{2} | p \rangle \langle p | x - \frac{s}{2} \rangle \langle x - \frac{s}{2} | \Psi \rangle ds \quad (4)$$

The four Dirac brackets are read from right to left as follows: (1) is the amplitude that a particle in the state Ψ has position $(x - s/2)$; (2) is the amplitude that a particle with position $(x - s/2)$ has momentum p ; (3) is the amplitude that a particle with momentum p has position $(x + s/2)$; (4) is the amplitude that a particle with position $(x + s/2)$ is (still) in the state Ψ . Thus, in Dirac notation the integrand is the quantum equivalent of a classical phase-space trajectory for a quantum system in the state Ψ .

Integration over s creates a superposition of all possible quantum trajectories of the state Ψ , which interfere constructively and destructively, providing a quasi-probability distribution in phase space. As an example, the Wigner probability distribution for a double-slit experiment is shown in the figure below [14, 27]. The oscillating positive and negative values in the middle of the Wigner distribution signify the interference associated with a quantum superposition, distinguishing it from a classical phase-space probability distribution. In the words of Leibfried et al. [14], the Wigner function is a “mathematical construct for visualizing quantum trajectories in phase space.”



Wigner distribution function for the double-slit experiment.

The calculation of the Wigner double-slit distribution function is given in the following tutorial.

<http://www.users.csbsju.edu/~frioux/wigner/DBL-SLIT-NEW.pdf>

Computational details for the generation of the Wigner distribution for the particle-in-the-box and the harmonic oscillator are also available.

<http://www.users.csbsju.edu/~frioux/wigner/wigner-pib.pdf>

<http://www.users.csbsju.edu/~frioux/wigner/WignerSHO.pdf>

Phase-space quantum mechanical calculations using the Wigner distribution are compared with coordinate- and momentum-space calculations in the following tutorial.

<http://www.users.csbsju.edu/~frioux/wigner/CoordMomPhase.pdf>

Literature cited:

- [1] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.* **40**, 749 – 759 (1932).
- [2] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, "Distribution functions in physics: Fundamentals," *Phys. Rep.* **106**, 121 – 167 (1984).
- [3] Y. S. Kim and E. P. Wigner, "Canonical transformations in quantum mechanics," *Am. J. Phys.* **58**, 439 – 448 (1990).
- [4] J. Snygg, "Wave functions rotated in phase space," *Am. J. Phys.* **45**, 58 – 60 (1977).
- [5] N. Mukunda, "Wigner distribution for angle coordinates in quantum mechanics," *Am. J. Phys.* **47**, 192 – 187 (1979).
- [6] S. Stenholm, "The Wigner function: I. The physical interpretation," *Eur. J. Phys.* **1**, 244 – 248 (1980).
- [7] G. Mourgues, J. C. Andrieux, and M. R. Feix, "Solutions of the Schrödinger equation for a system excited by a time Dirac pulse of pulse of potential. An example of the connection with the classical limit through a particular smoothing of the Wigner function," *Eur. J. Phys.* **5**, 112 – 118 (1984).
- [8] M. Casas, H. Krivine, and J. Martorell, "On the Wigner transforms of some simple systems and their semiclassical interpretations," *Eur. J. Phys.* **12**, 105 – 111 (1991).
- [9] R. A. Campos, "Correlation coefficient for incompatible observables of the quantum mechanical harmonic oscillator," *Am. J. Phys.* **66**, 712 – 718 (1998).
- [10] H-W Lee, "Spreading of a free wave packet," *Am. J. Phys.* **50**, 438 – 440 (1982).
- [11] D. Home and S. Sengupta, "Classical limit of quantum mechanics," *Am. J. Phys.* **51**, 265 – 267 (1983).

- [12] W. H. Zurek, “Decoherence and the transition from quantum to classical,” *Phys. Today* **44**, 36 – 44 (October 1991).
- [13] M. C. Teich and B. E. A. Saleh, “Squeezed and antibunched light,” *Phys. Today* **43**, 26 – 34 (June 1990).
- [14] D. Leibfried, T. Pfau, and C. Monroe, “Shadows and mirrors: Reconstructing quantum states of motion,” *Phys. Today* **51**, 22 – 28 (April 1998).
- [15] W. P. Schleich and G. Süssmann, “A jump shot at the Wigner distribution,” *Phys. Today* **44**, 146 – 147 (October 1991).
- [16] R. A. Campos, “Correlation coefficient for incompatible observables of the quantum harmonic oscillator,” *Am. J. Phys.* **66**, 712 – 718 (1998).
- [17] R. A. Campos, “Wigner quasiprobability distribution for quantum superpositions of coherent states, a Comment on ‘Correlation coefficient for incompatible observables of the quantum harmonic oscillator,’” *Am. J. Phys.* **67**, 641 – 642 (1999).
- [18] C. C. Gerry and P. L. Knight, “Quantum superpositions and Schrödinger cat states in quantum optics,” *Am. J. Phys.* **65**, 964 – 974 (1997).
- [19] K. Ekert and P. L. Knight, “Correlations and squeezing of two-mode oscillations,” *Am. J. Phys.* **57**, 692 – 697 (1989).
- [20] P. J. Price, “Quantum hydrodynamics and virial theorems,” *Am. J. Phys.* **64**, 446 – 448 (1995).
- [21] M. G. Raymer, “Measuring the quantum mechanical wave function,” *Contemp. Phys.* **38**, 343 – 355 (1997).
- [22] H-W Lee, A. Zynarski, and P. Kerr, “One-dimensional scattering by a locally periodic potential,” *Am. J. Phys.* **57**, 729 – 734 (1989).
- [23] A. Royer, “Why are the energy levels of the quantum harmonic oscillator equally spaced?” *Am. J. Phys.* **64**, 1393 – 1399 (1996).
- [24] D. F. Styer, et al., “Nine formulations of quantum mechanics,” *Am. J. Phys.* **70**, 288 – 297 (2002).
- [25] M. Belloni, M. A. Doncheski, and R. W. Robinett, “Wigner quasi-probability distribution for the infinite square well: Energy eigenstates and time-dependent wave packets,” *Am. J. Phys.* **72**, 1183 – 1192 (2004).
- [26] W. B. Case, “Wigner functions and Weyl transforms for pedestrians,” *Am. J. Phys.* **76**, 937 – 946 (2008).
- [27] Ch. Kurtsiefer, T. Pfau, and J. Mlynek, “Measurement of the Wigner function of an ensemble of helium atoms,” *Nature* **386**, 150 – 153 (1997).
- [28] W. H. Zurek, “Sub-Planck structure in phase space and its relevance for quantum decoherence,” *Nature* **412**, 712 – 717 (2001).