Introduction

Chaotic systems in mathematics and physics are systems for which solutions are possible, but the solutions are heavily dependent on the initial conditions. The weather is a famous example of a chaotic system. Small changes in conditions can cause large differences in later weather, which is the root of the difficulty in forecasting [Wikipedia Contributors, 2009a].

In this exercise we will examine chaos in a double pendulum, which is a much simpler system than the weather. A double pendulum is built from one pendulum attached to the base of another pendulum [Rafat et al., 2009]. The double pendulum in Figure 1 consists of mass $m_1$ attached to the end of a string of length $l_1$. A second pendulum of mass $m_2$ with a string of length $l_2$ is attached to the first mass. The location of the masses is described using the variables $\theta_1$ and $\theta_2$.

Following this notation, the equations of motion for the double pendulum are [Shinbrot et al., 1992; Wikipedia Contributors, 2009b]:

\[
\ddot{\theta}_1 = \frac{g(\sin \theta_2 \cos(\Delta \theta) - \mu \sin \theta_1) - (l_2 \dot{\theta}_2^2 + l_1 \dot{\theta}_1^2 \cos(\Delta \theta)) \sin(\Delta \theta)}{l_1(\mu - \cos^2(\Delta \theta))}
\]

\[
\ddot{\theta}_2 = \frac{g\mu(\sin \theta_1 \cos(\Delta \theta) - \sin \theta_2) + (\mu l_1 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 \cos(\Delta \theta)) \sin(\Delta \theta)}{l_2(\mu - \cos^2(\Delta \theta))}
\]

where $g$ is the gravitational acceleration which equals 9.8 m/s$^2$ near Earth’s surface, $\mu = -1 + \frac{m_1}{m_2}$ and $\Delta \theta = \theta_1 - \theta_2$ [Shinbrot et al., 1992].

If the angles are small these expressions simplify to:

\[
\dot{\theta}_1 \approx \frac{g(\theta_2 - \mu \theta_1)}{l_1(\mu - 1)}
\]

\[
\dot{\theta}_2 \approx \frac{g\mu(\theta_1 - \theta_2)}{l_2(\mu - 1)}
\]

The results presented above are for a simplified version of a double pendulum where all of the mass is concentrated in a ball at the end of a massless rod for each segment of the
pendulum. The double pendulum that you will be using here is called a compound double pendulum, and it is made with three bars of uniform cross-section. The top segment is made with two bars and they attach to the bottom bar with bearings.

The differential equations for this compound pendulum are a little different from those in the literature [Shinbrot et al. 1992] [Wikipedia Contributors 2009b] because this compound pendulum has \( m_1 \approx 2m_2 \). For simplicity in the equations below, we will take \( m_1 = 2m_2 = 2m \) and \( l_1 + l_2 = l \). In that case:

\[
\ddot{\theta}_1 = \left( -\frac{3}{2} \right) \frac{6\left(\frac{g}{l}\right)(2 \sin \theta_1 - \cos (\Delta \theta) \sin \theta_2) + 4 \sin (\Delta \theta) \dot{\theta}_2^2 + 3 \sin (2\Delta \theta) \dot{\theta}_1^2}{20 - 9 \cos^2 (\Delta \theta)}
\]

\[
\ddot{\theta}_2 = \left( \frac{3}{2} \right) \frac{\left(\frac{g}{l}\right)(-10 \sin \theta_2 + 18 \cos (\Delta \theta) \sin \theta_1) + 20 \sin (\Delta \theta) \dot{\theta}_1^2 + 3 \sin (2\Delta \theta) \dot{\theta}_2^2}{20 - 9 \cos^2 (\Delta \theta)}
\]

In the small angle approximation, these equations simplify down to:

\[
\ddot{\theta}_1 \approx \frac{18g(\theta_2 - 2\theta_1)}{22l}
\]

\[
\ddot{\theta}_2 \approx \frac{3g(9\theta_1 - 5\theta_2)}{11l}
\]

**Explorations**

In this exercise you want to compare mathematical results for the double pendulum to your observations of a real double pendulum.
Mathematical

Mathematica Simulation of Double Pendulum

Start by playing with a Mathematica simulation of a simple double pendulum [Morris 2009]. Try different initial conditions and observe the range of behavior that you can get out. Experiment with the behavior for both large and small initial angles.

Then see if you can get any normal mode behavior with \( l_1 = l_2 \) and \( m_1 = m_2 \). A normal mode is a state of the system where both pieces move sinusoidally with the same frequency. Normal modes are cases of relatively simple motion. In one normal mode \( \theta_1 = \theta_2 \) and the double pendulum oscillates like it is one long pendulum. In the other normal mode, \( \theta_1 = -\theta_2 \) and the bottom of the double pendulum stays directly under the top of the pendulum. Experiment with the behavior for both large and small initial angles.

Modeling a Compound Double Pendulum

Now you will model the behavior of the compound double pendulum that we are using below. You may want to jump back and forth between making the observations of the real double pendulum described below and looking at its modeled behavior.

Find the relevant dimensions for the double pendulum that you will need for your calculations. Use those measurements along with the differential equations given above for the compound double pendulum and whatever mathematical tools you like to solve for the behavior of the compound double pendulum. Compare the experimental and theoretical results. You can use angle versus time plots or some other type of visualization that you prefer for your comparisons.

Physical

Recording video

In your problem solution you may want to make use of the provided webcams to record the behavior of the double pendulum. You can use the program xawtv to monitor the live feed of the webcam. The program mencoder can be used for recording movies of your results you may want to use a command of the form:

\[
\text{mencoder} \; \text{tv://} \; \text{-tv} \; \text{driver=v4l2}:\text{width}=640:\text{height}=480:fps=60:device=/dev/video0 \; \text{\textbackslash \textbackslash} \; \text{-nosound} \; \text{-ovc lavc} \; \text{-lavcopts vcodec=mjpeg} \; \text{-o} \; \text{test.avi}
\]

To stop the recording hit Ctrl-Z in the terminal window that you ran mencoder in. Then kill the mencoder process by using ps to find the process number for the mencoder process, and killing that process with a command of the form:

\[
\text{kill} \; -9 \; \text{PROCESS\_NUMBER}
\]

where you replace PROCESS\_NUMBER with the number that you found using ps.

To playback your videos you can use mplayer or another video program.
Small Angles

Explore the results for the small angle approximation. For the case when the \( m_1 = m_2 \) there are two normal modes of this motion for small angles. See if you can cause either normal mode behavior in this double pendulum. If you can, get a movie of the normal mode.

Large angles

For large initial angles, the motion should be chaotic. Experiment with different initial conditions that you think are reproducible. See if you can get reproducible results, or if the results are chaotic. Make use of the webcams for taking your data.

Classify the results that you do get. What sort of behavior do you get from the rods of the pendulum? Is the behavior repetitive or consistent in anyway? How long does the pendulum continue to oscillate for a given set of initial conditions? Does the time length of the oscillations vary in an understandable way for different initial conditions?

Can you get normal mode or other regular behavior for large angles? Describe the behavior.

For all of these cases compare your results to your numerical calculations above.

Large initial velocities

In the above discussion, it was assumed that you started the pendulum with no initial speed. If you start the pendulum with a high enough initial speed, you should be able to get another type of reproducible motion. Try to get this motion and describe what it looks like.

References


