

Graver Complexity of Monomial

Curves in \mathbb{P}^3

Kristen Ann Nairn

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate Schools of Arts and Sciences

COLUMBIA UNIVERSITY

2003

© 2003

Kristen A. Nairn

All Rights Reserved

ABSTRACT

Graver Complexity of Monomial

Curves in \mathbb{P}^3

Kristen A. Nairn

The Graver complexity of a matrix \mathcal{A} is defined in terms of higher Lawrence liftings [18]. We investigate the geometric properties of the Graver basis of matrices that define monomial curves in \mathbb{P}^3 and use it to determine an upper bound for the Graver complexity of the matrix. We apply the results of Barvinok [1] to define a rational generating function for the Graver basis of \mathcal{A} . Bayer-Popescu-Sturmfels [2] showed that the diagonal embedding of a unimodular toric variety X has as its defining ideal a Lawrence ideal determined by elements in the Chow group $CL(X)$. We extend this result to define the diagonal embedding in terms of higher Lawrence liftings.

Contents

1	Introduction	1
1.1	Graver Complexity of Monomial Curves	7
1.2	Counting Lattice Points	9
2	Diagonal Embeddings of Toric Varieties	10
2.1	The Audin-Cox Homogeneous Coordinate Ring	10
2.2	Lawrence Ideals	12
3	Circuits	19
4	Algebraic and Geometric Results for matrices $\mathcal{A} = (0, a, b, a + b)$	24
5	Coverings By Circuits	31
6	Structure and Generating Functions of codim 2 Graver Bases	39
6.1	Computing the Hilbert Basis of a 2-dimensional Cone	40
6.2	Constructing the Generating Function of $Gr(\mathcal{A})$	43
7	$2c$ Conjecture for B_c	45
7.1	Primitive Partition Identities	48
8	The Graver Complexity of a matrix $\mathcal{A} \subset \mathbb{Z}^{2 \times 4}$	57
9	Complete Bipartite Graphs	67

10 Tables	74
10.1 General Matrices $\mathcal{A} = (0, i_2, i_3, i_4)$	74
10.2 Matrices $\mathcal{A} = (0, a, b, a + b)$	78
10.3 Matrices $\mathcal{A} = (0, a, b, ab)$	79
10.4 Other Special forms of the matrix \mathcal{A}	80
10.5 Integer Programming Relationship	81

Acknowledgements

I would like to thank my advisor Dave Bayer for insightful conversations. I am grateful to Henry Pinkham for guidance and I would like to thank Agnes Szilard for her support. I am indebted to Bernd Sturmfels who was truly an inspiration and directed my thesis. And I thank Sorin Popescu for his generosity and math discussions.

Thanks goes to Brian Greene who provided an oasis of tranquility in a sea of chaos. I would also like to thank John Morgan; without his encouragement, dedication and support, I would not be in graduate school.

I would like to thank Raymond Hemmecke for all the math discussions and personal motivation. A heartfelt thank you goes to my posse: Hal Schenk, Adam McInroy, Jeff Phan, Jeff Acter and Marie-Amelie Bertin. None of this would have been possible without their generosity and sense of humor.

I include a special thanks to Dolores, Laurent, Mary and Terrance for all their assistance. And thanks to those at MSRI, especially Michael Singer, Max, Kassia, Silvio and Rachelle.

Lastly, I would like to thank my family and friends who believed in me and cheered me on. And I give a special thank you to my muse, Harry.

1 Introduction

The beauty of studying Graver bases is that they can be viewed from many aspects of mathematics. Graver bases first appeared as a universal test set for integer programming problems, [9]. Since that time, they have been utilized for counting lattice points of polyhedra, finding the Hilbert basis of a given cone, they are related to the transportation problem and the knapsack problem. We wish to add to the already vast body of knowledge related to Graver bases.

The original motivation for this thesis comes from a paper of Santos-Sturmfels [18]. Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ be a $d \times n$ integer vector configuration where $a_i \in \mathbb{Z}^d$. They apply the notion of higher Lawrence liftings to $r_1 \times r_2$ **contingency tables**; these are arrays of nonnegative real numbers, where the r_1 and r_2 are fixed explanatory variables. In toric algebra, this table is a model represented by a sparse, unimodular vector configuration $\mathcal{A} \subseteq \mathbb{Z}^{d \times n}$ where $d = r_1 + r_2$ and $n = r_1 r_2$. Given such a table, we can compute the **marginal** which is a vector encoding the row and column sum of the entries in the table. Combining k contingency tables defines an $r_1 \times r_2 \times k$ contingency table where k is allowed to vary. An interesting question is to find a minimal k such that the Graver basis stabilizes up to symmetry. This integer k is the Graver complexity of the matrix \mathcal{A} . To arrive at the Graver complexity, we need the notion of a higher Lawrence lifting.

Definition 1.1. *Let $\mathcal{A} = \{a_1, \dots, a_n\} \subseteq \mathbb{Z}^{d \times n}$ be a vector configuration. Introduce an hierarchy of lifts $\mathcal{A}^{(2)}, \mathcal{A}^{(3)}, \mathcal{A}^{(4)}, \dots$ where the **r-th Lawrence lifting** of \mathcal{A} , denoted*

by $\mathcal{A}^{(r)}$ consists of $r \cdot n$ vectors in \mathbb{Z}^{dr+n} written as

$$\mathcal{A}^{(r)} = \{(a_i \otimes e_j) \oplus e_i \mid 1 \leq i \leq n, 1 \leq j \leq r\}$$

where e_i, e_j are the standard vectors in $\mathbb{Z}^n, \mathbb{Z}^r$ respectively.

The lattice of linear relations on $\mathcal{A}^{(r)}$ is

$$\mathcal{L}(\mathcal{A}^{(r)}) = \{u^{(1)}, \dots, u^{(r)} \in (\mathbb{Z}^n)^r : u^{(i)} \in \mathcal{L}(\mathcal{A}) \forall i, \sum u^{(i)} = 0\}.$$

The elements of $\mathcal{L}(\mathcal{A}^{(r)})$ can be thought of as integer $r \times n$ tables whose column sums are zero and whose \mathcal{A} -weighted row sums are zero. The **type** of such a table is the number of non-zero vectors $u^{(i)}$.

Let k be a field and $\mathcal{A} = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{Z}^{d \times n}$. Identify each vector a_i with a monomial t^{a_i} in the Laurent polynomial ring $k[t^\pm] = k[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$. The kernel of the map

$$\phi: k[x_1, x_2, \dots, x_n] \rightarrow k[t^\pm] \quad \text{by } x_i \mapsto t^{a_i}$$

defines the **toric ideal** $I_{\mathcal{A}}$ of \mathcal{A} . It is the prime binomial ideal

$$I_{\mathcal{A}} = \langle \mathbf{x}^\alpha - \mathbf{x}^\beta : \alpha, \beta \in \mathbb{N}^n, \alpha - \beta \in \mathcal{L}(\mathcal{A}) \rangle.$$

Naturally associated to $\mathcal{A}^{(r)}$ is an hierarchy of lattice basis ideals $I_{\mathcal{A}^2}, I_{\mathcal{A}^3}, \dots$. A **Markov basis** of $\mathcal{A}^{(r)}$ is the minimal set of generators of its toric ideal $I_{\mathcal{A}^{(r)}}$. The following result says the Markov basis stabilizes for some $r \gg 0$:

Theorem 1.2. (*Santos, Sturmfels*)

For any vector configuration $\mathcal{A} = \{a_1, \dots, a_n\}$ with $a_i \in \mathbb{Z}^d$, there exists a constant

$m = m(\mathcal{A})$ such that every higher Lawrence lifting $\mathcal{A}^{(r)}$ has a Markov basis consisting of tables having type at most m .

Definition 1.3. The minimum value $m(\mathcal{A})$ is called the **Markov complexity**.

Definition 1.4. Given a vector $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$, the **1-norm** of \mathbf{u} is $\|\mathbf{u}\|_1 = \sum_{i=1}^n |u_i|$.

Circuits play a crucial role in understanding the geometric structure of the Graver basis of a matrix.

Definition 1.5. For $\mathbf{v} \in \mathbb{R}^n$, the support of \mathbf{v} is $\text{supp}(\mathbf{v}) = \{i: v_i \neq 0\}$. A vector $\mathbf{u} \in \ker_{\mathcal{A}}$ is called a **circuit** of \mathcal{A} if $\text{supp}(\mathbf{u})$ is minimal with respect to inclusion and the coordinates of u are relatively prime.

We may define the 1-norm of a nonzero vector $\mathbf{u} \in \ker(\mathcal{A})$ by applying Cramer's rule to the $d \times (d + 1)$ submatrices of \mathcal{A} :

$$\|\mathbf{u}\|_1 = \sum_{j=1}^{d+1} (-1)^j | \det(a_{i_1}, \dots, a_{i_{j-1}}, a_{i_{j+1}}, \dots, a_{i_{d+1}}) | \mathbf{e}_{i_j}$$

where \mathbf{e}_{i_j} is the i -th unit vector. Circuits defined by Cramer's rule are called **true circuits**. Denote by $\mathcal{C}(\mathcal{A})$ the set of all circuits of a $d \times n$ integer matrix \mathcal{A} .

Definition 1.6. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, Define the relation \sqsubseteq on \mathbb{R}^n by $\mathbf{u} \sqsubseteq \mathbf{v}$ if $u^{(i)}v^{(i)} \geq 0$ and $|u^{(j)}| \leq |v^{(j)}|$ for every component $1 \leq j \leq n$. We say that \mathbf{u} **reduces** \mathbf{v} if $\mathbf{u} \sqsubseteq \mathbf{v}$.

The Graver basis can be defined in terms of the Hilbert basis of pointed, convex rational polyhedral cones. An n -dimensional **cone** in \mathbb{R}^m is a nonempty set of vectors $C \subseteq \mathbb{R}^m$ that with any finite set of vectors also contains all their linear combinations with nonnegative coefficients. Write

$$C = \text{cone}(a_1, a_2, \dots, a_n) = \{\lambda_1 a_1 + \dots + \lambda_n a_n : \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

Definition 1.7. Let $\ker_{\mathbb{Z}}(A) = \{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} = 0\}$ be the kernel of the $d \times n$ matrix A over \mathbb{Z} and let O_{ρ} denote an orthant in \mathbb{R}^n where $\rho \in \{+, -\}^n$. Let $C_{\rho} = \ker(A) \cap O_{\rho}$ denote the cone. If, in addition, $\{0\}$ is the largest and only linear subspace of \mathbb{R}^n that is contained in C_{ρ} , then C_{ρ} is a **pointed polyhedral cone**. Such a cone can be described by a system of inequalities $C = \{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} \leq 0\}$ with a suitable matrix $A \in \mathbb{Z}^{d \times n}$.

Definition 1.8. Given a nonzero pointed polyhedral cone C_{ρ} with $\rho \in \{+, -\}^n$, a subset of integral vectors $\mathcal{H}_{\rho} \subset C_{\rho}$ is a **Hilbert basis** of C_{ρ} if for every element $z \in C_{\rho}$, there exist nonnegative $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{Z}$ such that $z = \sum_{i=1}^s \alpha_i h_i$ and \mathcal{H}_{ρ} has minimal cardinality with respect to all such subsets of C_{ρ} .

Definition 1.9. The **Graver basis** of a matrix \mathcal{A}

$$\mathcal{G}r(\mathcal{A}) = \bigcup_{\rho} \mathcal{H}_{\rho} \setminus 0$$

is the set of nonzero minimal elements in the poset $\{\mathcal{H}_{\rho}, \sqsubseteq\}$ for each cone C_{ρ} .

A technique to compute the Graver basis of a matrix \mathcal{A} is to use the Lawrence lifting.

Definition 1.10. Let $\mathcal{A} \subseteq \mathbb{Z}^{d \times n}$ and define the **Lawrence lifting** $\Lambda(\mathcal{A})$ of the matrix to be the $(d + n) \times 2n$ integer matrix

$$\Lambda(\mathcal{A}) = \begin{pmatrix} \mathcal{A} & 0 \\ I & I \end{pmatrix},$$

where I is the $n \times n$ identity matrix.

The following is a well-known result:

Theorem 1.11. *Sturmfels [21]*

For a Lawrence type matrix $\Lambda(\mathcal{A})$ the following sets of binomials coincide:

1. The Graver basis of $\Lambda(\mathcal{A})$
2. The universal Gröbner basis of $\Lambda(\mathcal{A})$
3. Any reduced Gröbner basis of $\Lambda(\mathcal{A})$
4. Any minimal generating set of $I_{\Lambda(\mathcal{A})}$

We now define the Graver complexity in terms of tables:

Definition 1.12. The **Graver complexity**, or **Graver degree**, $g(\mathcal{A})$, of a matrix \mathcal{A} is defined to be the maximum type of any table that appears in the Graver basis of some higher Lawrence lifting $\mathcal{A}^{(r)}$.

We state the circuit complexity for reference.

Definition 1.13. *The circuit complexity $c(\mathcal{A})$ is the maximal type of any table that is a circuit of some higher Lawrence lift $\mathcal{A}^{(r)}$.*

Given the aforementioned complexities of a matrix \mathcal{A} we have the following:

- By the natural inclusion of circuits in the Graver basis, $\mathcal{C}(\mathcal{A}^{(r)}) \subseteq \mathcal{Gr}(\mathcal{A}^{(r)})$ and thus the circuit complexity satisfies $c(\mathcal{A}) \leq g(\mathcal{A})$.
- The Markov complexity also relates to the Graver complexity by $m(\mathcal{A}) \leq g(\mathcal{A})$ [18]

Thus the Graver complexity determines an upper bound and is therefore of interest to study. For computations of the Graver complexity, we utilize the extremely powerful theorem

Theorem 1.14. *Santos,Sturmfels [18]*

The Graver complexity $g(\mathcal{A})$ of a vector configuration $\mathcal{A} = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{Z}^{d \times n}$ is the maximum 1-norm of the elements in the Graver basis of the Graver basis of \mathcal{A} .

To compute the Graver basis of the Graver basis of \mathcal{A} , transpose the elements in $\mathcal{Gr}(\mathcal{A})$ into column vectors of a new matrix $\mathcal{Gr}(\mathcal{A})^T$ and take the Graver basis of this new matrix. A vector $\psi \in \mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$ of maximal 1-norm is called the **Graver representative**.

There is a natural relationship between the $m \times n$ contingency tables and the complete bipartite graph $K_{m,n}$. Using 4ti2 [11], the Graver complexity of the complete bipartite graph $K_{3,3}$ is $g(3 \times 3) = 9$. For the $K_{3,4}$ graph, however, 4ti2 was unable to complete the $\mathcal{Gr}(\mathcal{Gr}(3 \times 4))$ computation. By considering different linear sections of

the matrix $\mathcal{G}r(\mathcal{A})^T$, we found the following (non-unique) Graver representative

$$(0, -3, 3, 0, 0, 0, 4, 0, -4, 0, 0, 0, 0, 0, 0, 5, -6, 2).$$

Thus, we state the following

Observation 1.15. *The Graver complexity of the matrix associated to the complete bipartite graph $K_{3,4}$ is $g(3 \times 4) \geq 27$.*

We will prove some complexity results for more simple complete bipartite graphs and note some underlying structure in the Graver bases for the complete bipartite graphs $K_{3,3}$ and $K_{3,4}$.

1.1 Graver Complexity of Monomial Curves

We focus on integer matrices of the form $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i_1 & i_2 & i_3 & i_4 \end{pmatrix}$ with $0 \leq i_1 < i_2 < i_3 < i_4$ because of ease in computing the Graver complexity and because the projective variety defined by the kernel of this matrix is a monomial curve in \mathbb{P}^3 [21]. Rational normal curves in k^n are given as the image of the polynomial parameterization

$$\phi: k \rightarrow k^n \quad \text{by} \quad \phi(t) = (t, t^2, t^3, \dots, t^n).$$

The projective closures of these affine varieties are rational normal curves in \mathbb{P}^n . These projective varieties are defined by the set of homogeneous quadrics obtained by taking all possible 2×2 subdeterminants of the $2 \times n$ matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

In general, it is not clear how the degree of the generators of the homogeneous ideal $I_{\mathcal{A}}$ are related to the matrix \mathcal{A} . L'vovsky [14] shows that the toric ideal $I_{\mathcal{A}}$

defined by a $2 \times n$ integer matrix $\mathcal{A} = \{(1, i_1), (1, i_2), \dots, (1, i_{n-1}), (1, i_n)\}$ with $0 \leq i_1 < i_2 < \dots < i_n$ is generated by elements of degree at most the sum of the two largest consecutive differences $i_k - i_{k-1}$. Thus, if $\delta_k = i_k - i_{k-1}$, where $1 \leq k \leq n$, then the maximal degree of the generators for the monomial curve ideal $I_{\mathcal{A}}$ is $\max\{\delta_k + \delta_j\}$ for $1 \leq k < j \leq n$. This is the regularity of the ideal $I_{\mathcal{A}}$. Our main theorem states that the Graver complexity of 2×4 integer matrices representing monomial curves can be bounded above by the integers defining the matrix \mathcal{A} itself.

The first special case we consider are 2×4 matrices of the form $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & a+b \end{pmatrix}$ where $a < b$. The Graver basis $\mathcal{G}r(\mathcal{A})$ is seen geometrically as a line and the Graver basis of the Graver basis of \mathcal{A} has kernel isomorphic to the the kernel of the matrix

$$B_c = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & 2 & \dots & c & 1 \end{pmatrix},$$

where $c = a + b$.

Denote by $\maxg(\mathcal{A}) = \max\{\|u\|_1 : u \in \mathcal{G}r(\mathcal{A})\}$ the elements in the Graver basis of \mathcal{A} with maximal 1-norm.

Theorem 1.16. *Hosten, 2.3.3*

Let \mathcal{A} be a $d \times n$ integer matrix with $\text{rank}(\mathcal{A}) = d$ and assume without loss of generality that $D(\mathcal{A}) = |\det(a_1, a_2, \dots, a_n)|$. Then $\maxg(\mathcal{A}) \leq 2^d(d+1)^d D(\mathcal{A})$.

Thus for a $2 \times n$ matrix \mathcal{A} , $\maxg(\mathcal{A}) \leq 36D(\mathcal{A})$. Hosten ([12] Corollary 2.3.5) improved this bound to

$$\maxg(\mathcal{A}) \leq 12D(\mathcal{A}).$$

Remark 1.17. *For the matrix B_c , Hosten's result states that $\maxg(B_c) \leq 12c$.*

As a result of computations in *4ti2* for $1 \leq c \leq 18$, we make the following conjecture:

Conjecture 1.18. *2c Conjecture*

For the matrix B_c with $c \geq 3$, the Graver complexity is $g(B_c) = 2c$.

1.2 Counting Lattice Points

The notion of counting lattice points inside polyhedra and even merely detecting whether there is a lattice point in polyhedra (see [19]) is a long standing problem. Lenstra, [13], was the first to give a polynomial time algorithm for problems in fixed dimensions. Barvinok [1] used Lenstra's algorithm along with Brion's theorem [3] (which counts lattice points in convex rational pointed cones) to create a new algorithm that produces a *rational* generating function for counting lattice points in polynomial time when the dimension of the polytope is fixed. The computer program LattE [6] is the first to implement Barvinok's algorithm. The structure of the Graver basis of the 2×4 matrices can be used to define a rational generating function, thus showing that the Graver basis is a collection of polynomially many line segments. We use continued fractions, see [8] or [16], to find the Hilbert basis in a convex pointed cone. Thus our polynomial size description of the Graver basis as a rational generating function relates to the recent work of Barvinok. See Firla [7] for an integer programming interpretation.

2 Diagonal Embeddings of Toric Varieties

In this section, we consider unimodular toric varieties X in an homogeneous coordinate ring $S(\Sigma) = k[x_1, x_2, \dots, x_n]$ over k in variables x_1, \dots, x_n and Σ is a fan in $N \cong \mathbb{Z}^n$ [5]. We will use the notion of a higher Lawrence lifting to extend a result of Bayer-Popescu-Sturmfels [2] that embeds a unimodular toric variety $X \hookrightarrow X \times X$.

2.1 The Audin-Cox Homogeneous Coordinate Ring

Let X be a toric variety defined by a fan Σ in $N \cong \mathbb{Z}^n$. The algebraic torus $T = \text{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$ acts on X . One-dimensional cones of Σ form a set $\Sigma(1) = \{\rho_1, \rho_2, \dots, \rho_n\}$ and, for any cone $\sigma \in \Sigma$, let $\sigma(1) = \{\rho \in \Delta(1) \mid \rho \subset \sigma\}$ be the set of 1-dimensional faces of σ . Each element $\rho \in \Delta(1)$ corresponds to a T -invariant Weil divisor D_ρ in the toric variety X . Thus $D = \sum_{\rho} a_{\rho} D_{\rho}$ is an element of the free abelian group $\mathbb{Z}^{\Sigma(1)}$ defined by the T -invariant Weil divisors. If $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the \mathbb{Z} -dual of N , then every $m \in M$ gives a character $\chi^m: T \rightarrow \mathbb{C}^*$ and determines a Cartier divisor $\text{div}(\chi^m): = D_m = \sum_{\rho} \langle m, n_{\rho} \rangle D_{\rho}$. Because $\Delta(1)$ spans $N \otimes_{\mathbb{Z}} \mathbb{R}$, the rational function χ^m on X defines an injective map

$$M \rightarrow \mathbb{Z}^{\Sigma(1)} \quad \text{by } m \mapsto D_m.$$

Thus there is the commutative diagram, by Fulton ([8] section 3.4)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \text{Div}_T(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array}$$

Let $\text{Pic}(X)$ be the group of all line bundles modulo isomorphism. For any irreducible variety X , the map $D \mapsto \mathcal{O}(D)$ defines a homomorphism from the group of

Cartier divisors on X onto $\text{Pic}(X)$ whose kernel is the group of principal divisors. The group A_{n-1} consists of all Weil divisors modulo the subgroup of divisors of rational functions. Since the toric variety X is normal, the map $\text{Pic}(X) \hookrightarrow A_{n-1}$ is an embedding sending $D \mapsto [D]$. Therefore, a divisor $D \in \mathbb{Z}^{\Sigma(1)}$ determines an element $\alpha = [D]$ in the **Chow group** $A_{n-1}(X)$ of the variety X , where $n = \dim(X)$.

To each element $\rho \in \Delta(1)$, assign a variable x_ρ and consider the polynomial ring

$$S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)].$$

Every monomial $x^D = \prod_\rho x_\rho^{a_\rho}$ determines a divisor $D = \sum_i a_i D_i$. The grading on S is given by the **degree** $\deg(x^D) = [D] \in A_{n-1}$ where any two monomials have the same degree if and only if they differ by an element of the torus. Let

$$S_\alpha = \bigoplus_{\deg(x^D)=\alpha} \mathbb{C} \cdot x^D$$

and define S as the **Audin-Cox homogeneous coordinate ring** of the toric variety X where

$$S = \bigoplus_{\alpha \in A_{n-1}(X)} S_\alpha,$$

with the property that $S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}$. This grading is homogeneous with respect to the degree grading.

For any cone $\sigma \in \Sigma$, let $\hat{\sigma} = \sum_{\rho \notin \sigma(1)} D_\rho$ be the divisor and let $\mathbf{x}^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ be the corresponding monomial. Then the ideal

$$B = \langle \mathbf{x}^{\hat{\sigma}} : \sigma \in \Sigma_{max} \rangle \subset S$$

is the ideal generated by $\mathbf{x}^{\hat{\sigma}}$ as σ ranges over all maximal cones of Σ . The zero set $Z = V(B)$ of this ideal describes the combinatorial structure of the fan Σ . Consider

X as the geometric quotient $X \cong (\mathbb{C}^{\Sigma(1)} - Z)/G$ where $G = \text{Hom}_{\mathbb{Z}}(A_{n-1}, \mathbb{C}^*)$ is the torus acting on the affine space $\mathbb{C}^{\Sigma(1)}$. Elements in $\mathbb{C}^{\Sigma(1)} - Z$ can be regarded as "homogeneous coordinates" for points in the toric variety X . This description defines X as a diagonal embedding into projective space. Given the complete fan $\Sigma \in \mathbb{Z}^m$ and the associated toric variety X , define the primitive generators of the one-dimensional cones of Σ as $b_1, \dots, b_n \in \mathbb{Z}^m$ and let B be the $n \times m$ matrix with row vectors b_i . Each of these b_i determines a T -invariant Weil divisor D_i on X . The short exact sequence

$$0 \rightarrow \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^n \xrightarrow{\pi} Cl(X) \rightarrow 0 \quad (1)$$

where π takes the i -th standard basis vector in \mathbb{Z}^n to the linear equivalence class $[D_i]$ of the corresponding divisor, defines the abelian group $Cl(X)$ of torus-invariant Weil divisors modulo linear equivalence called the **class group**. If the divisor class group is torsion free, then $Cl(X) = \mathbb{Z}^{n-m} = \mathbb{Z}^d$ and we may express π by a $d \times n$ matrix $A: \mathbb{Z}^n \rightarrow \mathbb{Z}^d$. If the rank $\text{rk}(A) = d$ then d elements in the kernel of A define the ideal I_A . Hence, we may define the homogeneous coordinate ring associated to the toric variety X by $R = k[x_1, \dots, x_n]$ with the grading given by the class group $Cl(X)$ via the morphism π in the short exact sequence (1).

2.2 Lawrence Ideals

Ziegler [22] describes the geometric construction of the Lawrence lifting for convex polytopes. We focus on an algebraic analogue of the Lawrence lifting by considering the defining ideals of unimodular toric subvarieties in a product of n projective lines $\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$. These defining ideals are binomial ideals in $2n$ variables called

Lawrence ideals and are of the form

$$J_{\mathcal{L}} = \langle \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}} \mid \mathbf{a} - \mathbf{b} \in \mathcal{L} \rangle \subset S = k[x_1, \dots, x_n, y_1, \dots, y_n],$$

where \mathcal{L} is a sublattice of \mathbb{Z}^n and k is a field. Write $(x_i: y_i)$ for the homogeneous coordinates of the i -th factor \mathbb{P}^1 and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. Because unimodular toric varieties are simplicial, we assume the toric variety X to be simplicial for the remainder of this section. For the non-simplicial case, see Mustața [15], Theorem 1.1.

We will use higher Lawrence liftings of arbitrary matrices \mathcal{A} to define Lawrence ideals in order to extend the following result from [2], Proposition 6.1:

Theorem 2.1. *The ideal $I_X \subset S$ defining the diagonal embedding $X \subset X \times X$ equals the Lawrence ideal $J_{\mathcal{L}}$ for the lattice $\mathcal{L} = \ker(\pi)$ of principal divisors.*

Proof. The toric variety $X \times X$ has the homogeneous coordinate ring

$$S = R \otimes_k R = k[x_1, \dots, x_n, y_1, \dots, y_n]$$

and the diagonal embedding $X \subset X \times X$ defines a closed subscheme that is represented by a $Cl(X) \times Cl(X)$ -graded ideal I_X in S , by Theorem 3.7 in [5]. This ideal I_X is the kernel of the map

$$\phi: S \rightarrow k[Cl(X)] \otimes R \quad \mathbf{x}^{\mathbf{u}}\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \mapsto [\mathbf{u}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}}.$$

Given the map (1), the ideal is

$$I_X = \langle \mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} - \mathbf{x}^{\mathbf{v}}\mathbf{y}^{\mathbf{u}} \mid \pi(\mathbf{u}) = \pi(\mathbf{v}) \text{ in } Cl(X) \rangle \subset S$$

and the result holds. \square

Corollary 2.2. *Let \mathcal{A} be a $d \times n$ integer matrix. Then the diagonal embedding of $X \hookrightarrow X \times X$ in Theorem 2.1 defining I_X is given by $\ker_{\mathbb{Z}} \Lambda(\mathcal{A})$.*

Proof. The Class group $Cl(X) = \bigoplus_i \mathbb{Z}[D_i]$ is a free abelian group, thus the map π can be expressed by the matrix \mathcal{A} . Elements in the kernel of the embedding $X \hookrightarrow X \times X$ will satisfy $a \mapsto (a, -a)$. Consequently, the embedding given in Theorem 2.1 is determined by the kernel of the Lawrence lifting $\Lambda(\mathcal{A})$. \square

Proposition 2.3. *Let $\mathcal{A} \subseteq \mathbb{Z}^{d \times n}$ be a vector configuration. The kernel of the Lawrence lifting $\Lambda(\mathcal{A})$ is isomorphic to that of the second higher Lawrence lifting $\mathcal{A}^{(2)}$.*

Proof. The defining ideal for $\Lambda(\mathcal{A})$ is generated by

$$\mathcal{L}(\Lambda(\mathcal{A})) = \langle (\mathbf{u}, -\mathbf{u}) \mid \mathbf{u} \in \ker_{\mathbb{Z}} \mathcal{A} \rangle.$$

By definition of higher Lawrence liftings,

$$\mathcal{A}^{(2)} = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{A} \\ I & I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathcal{A}\mathbf{u} \\ \mathcal{A}\mathbf{v} \\ \mathbf{u} + \mathbf{v} \end{pmatrix}.$$

Thus, $\mathbf{v} = -\mathbf{u}$ and $\mathbf{v} \in \ker_{\mathbb{Z}} \mathcal{A}$ which implies that $(u, -u)$ lies in the kernel of $\mathcal{A}^{(2)}$ for $\mathbf{u} \in \ker_{\mathbb{Z}} \mathcal{A}$. Therefore, the two matrices have isomorphic kernels. \square

Before we introduce a Lawrence ideal that defines the diagonal embedding of $X \hookrightarrow X \times X \times X$, we look at the ideal defined by elements in the kernel of $\mathcal{A}^{(3)}$ that correspond to tables of type 2.

Proposition 2.4. *Let \mathcal{A} be an integer $d \times n$ matrix and X a simplicial toric variety.*

Consider the map given by

$$\mathcal{A}^{(3)}: R^{\otimes 3} \rightarrow k[Cl(X)]^{\otimes 2} \otimes R \quad \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \otimes \mathbf{x}^{\mathbf{w}} \mapsto [\mathbf{u}] \otimes [\mathbf{v}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}+\mathbf{w}}.$$

The ideal I_{Δ} defining the small diagonal embedding of $\Delta \hookrightarrow X \times X \times X$ is a Lawrence ideal defined by elements in the kernel of $\mathcal{A}^{(3)}$ that correspond to tables of type 2.

Proof. The diagonal $\Delta \subset X \times X \times X$ in the Cox homogeneous coordinate ring $S = k[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]$ is given by sending $x \mapsto (x, x, x)$ for any $x \in X$.

The **small diagonals** in the embedding are defined as the set of elements

$$\{(p, p, p)\} = \cap \{(p, q, q) \cap (p, p, q) \cap (p, q, p)\}$$

which give the projection

$$X \times X \times X \rightarrow X \times X \xrightarrow{\Delta} X \quad \text{by} \quad (p, p, q) \mapsto (p, p) \mapsto p.$$

Let $\Delta = (a, a) \in X \times X$ denote the small diagonal in X^2 . Then set theoretically, the small diagonal in X^3 is the intersection of all the images of the small diagonals in X^2

$$\cap_{i < j} \text{im}(\Delta) \hookrightarrow X \times X \hookrightarrow X^3$$

Denote the images of Δ by

$$\Delta_{12} = (x, x, any) \quad \Delta_{13} = (x, any, x) \quad \Delta_{23} = (any, x, x).$$

This intersection of subvarieties corresponds to the sum of ideals

$$\begin{aligned} \Delta &= \Delta_{12} \cap \Delta_{13} \cap \Delta_{23} = V(I_{12}) \cap V(I_{13}) \cap V(I_{23}) \\ &= V(I_{12} + I_{13} + I_{23}) = V(I_{\Delta}) \end{aligned}$$

Thus the ideal defined by the small diagonal in X^3 is $I_{\Delta} = I_{12} + I_{13} + I_{23}$ where

$$I_{12} = \langle \mathbf{x}^{u^+} \mathbf{y}^{u^-} - \mathbf{x}^{u^-} \mathbf{y}^{u^+} : (\mathbf{u}, -\mathbf{u}, 0) \in \ker \mathcal{A}^{(3)}, \mathbf{u} = u^+ - u^- \in \ker \mathcal{A} \rangle$$

$$I_{13} = \langle \mathbf{x}^{v^+} \mathbf{z}^{v^-} - \mathbf{x}^{v^-} \mathbf{z}^{v^+} : (\mathbf{v}, 0, -\mathbf{v}) \in \ker \mathcal{A}^{(3)}, \mathbf{v} = v^+ - v^- \in \ker \mathcal{A} \rangle$$

$$I_{23} = \langle \mathbf{y}^{w^+} \mathbf{z}^{w^-} - \mathbf{y}^{w^-} \mathbf{z}^{w^+} : (0, \mathbf{w}, -\mathbf{w}) \in \ker \mathcal{A}^{(3)}, \mathbf{w} = w^+ - w^- \in \ker \mathcal{A} \rangle$$

Therefore, the Lawrence ideal of the small diagonal in X^3 is

$$I_{\Delta} = \langle \mathbf{x}^{u^+} \mathbf{y}^{u^-} - \mathbf{x}^{u^-} \mathbf{y}^{u^+}, \mathbf{x}^{v^+} \mathbf{z}^{v^-} - \mathbf{x}^{v^-} \mathbf{z}^{v^+}, \mathbf{y}^{w^+} \mathbf{z}^{w^-} - \mathbf{y}^{w^-} \mathbf{z}^{w^+} \rangle \subset S$$

satisfying $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \ker \mathcal{A}$ and $(\mathbf{u}, -\mathbf{u}, 0), (\mathbf{v}, 0, -\mathbf{v}), (0, \mathbf{w}, -\mathbf{w}) \in \ker \mathcal{A}^{(3)}$. By definition, these are precisely the elements in the kernel of $\mathcal{A}^{(3)}$ corresponding to tables of type 2. □

Corollary 2.5. *Let X be a simplicial toric variety. Consider the map*

$\mathcal{A}^{(n)}: R^{\otimes n} \rightarrow k[Cl(X)]^{\otimes n-1} \otimes R$ *given by*

$$\mathbf{x}_1^{\mathbf{u}_1} \otimes \mathbf{x}_2^{\mathbf{u}_2} \otimes \cdots \otimes \mathbf{x}_n^{\mathbf{u}_n} \mapsto [\mathbf{u}_1] \otimes [\mathbf{u}_2] \otimes \cdots \otimes [\mathbf{u}_{n-1}] \otimes \mathbf{x}^{\sum_i \mathbf{u}_i}.$$

The ideal $I_\Delta \subset S$ defining the small diagonal embedding $\Delta \hookrightarrow \underbrace{X \times X \times \cdots \times X}_n$ is a Lawrence ideal defined by elements in $\ker \mathcal{A}^{(n)}$ corresponding to tables of type 2.

Proof. The toric variety defined by the diagonal embedding of Δ has homogeneous coordinate ring

$$S = R^{\otimes n} = k[x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2, \dots, x_1^n, \dots, x_n^n]. \quad (2)$$

As in Proposition 2.4, the ideal generated by the small diagonal $\Delta \hookrightarrow X^n$ is the ideal $I_\Delta = \sum_{j=2}^n \sum_{i=1}^{n-1} I_{ij} \subset S$, namely

$$I_\Delta = \langle \mathbf{x}_1^{\mathbf{u}_1^+} \mathbf{x}_i^{\mathbf{u}_1^-} - \mathbf{x}_1^{\mathbf{u}_1^-} \mathbf{x}_i^{\mathbf{u}_1^+}, \mathbf{x}_2^{\mathbf{u}_2^+} \mathbf{x}_i^{\mathbf{u}_2^-} - \mathbf{x}_2^{\mathbf{u}_2^-} \mathbf{x}_i^{\mathbf{u}_2^+}, \dots, \mathbf{x}_{n-1}^{\mathbf{u}_{n-1}^+} \mathbf{x}_n^{\mathbf{u}_{n-1}^-} - \mathbf{x}_{n-1}^{\mathbf{u}_{n-1}^-} \mathbf{x}_n^{\mathbf{u}_{n-1}^+} \rangle$$

where $\mathbf{u}_i \in \ker \mathcal{A}$ and, for each i, j ,

$$I_{ij} = \langle \mathbf{x}_i^{\mathbf{u}_i^+} \mathbf{x}_j^{\mathbf{u}_i^-} - \mathbf{x}_i^{\mathbf{u}_i^-} \mathbf{x}_j^{\mathbf{u}_i^+} : (0, \dots, \mathbf{u}_i, 0, \dots, -\mathbf{u}_i, 0, \dots) \in \ker \mathcal{A}^{(n)} \rangle.$$

These are vectors in the kernel of $\mathcal{A}^{(n)}$ that correspond to tables of type 2. \square

In the homogeneous coordinate ring S , denote by I_{cox}^∞ the monomial ideal defined by the maximal cones in the fan Σ defining the toric variety X .

Theorem 2.6. *Let X be a simplicial toric variety and let*

$$\mathcal{A}^{(3)}: R^{\otimes 3} \rightarrow k[Cl(X)] \otimes k[Cl(X)] \otimes R \quad \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \otimes \mathbf{x}^{\mathbf{w}} \mapsto [\mathbf{u}] \otimes [\mathbf{v}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}+\mathbf{w}}$$

be the embedding of $X \subset X \times X \times X$, with $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \ker \mathcal{A}^{(3)}$. The defining ideal I_X of the embedding equals the Lawrence ideal $I_{\mathcal{L}}$ defined by the lattice $\mathcal{L} = \ker \mathcal{A}^{(3)}$.

Proof. The diagonal embedding $X \hookrightarrow X \times X \times X$ defines a closed subscheme that is represented by a $Cl(X) \times Cl(X) \times Cl(X)$ ideal $I_{\mathcal{A}^{(3)}}$ in the homogeneous coordinate ring

$$R \otimes_k R \otimes_k R = k[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]$$

of $X \subset X \times X \times X$. Denote by $J = I_{12} + I_{13} + I_{23}$ the ideal defining the small diagonal embedding of X into X^3 . To obtain the defining ideal of the embedding $X \hookrightarrow X^3$, saturate J by the irrelevant ideal I_{cox}^∞ . Then by the Nullstellensatz with respect to Cox, $I(V(J^{\text{sat}})) = I(V(J : I_{cox}^\infty)) = \sqrt{J^{\text{sat}}}$. Therefore, the ideal defining the diagonal embedding is $I_{\mathcal{L}} = \sqrt{(I_{12} + I_{13} + I_{23})^{\text{sat}}}$:

$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \ker \mathcal{A}^{(3)}$ if and only if $\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = 0$, $\mathbf{w}_i \in \ker \mathcal{A}$ with $\mathbf{w}_i = w_i^+ - w_i^-$.

These conditions define a Lawrence ideal

$$I_{\mathcal{L}} = \langle \mathbf{x}^{w_1^+} \mathbf{y}^{w_2^+} \mathbf{z}^{w_3^+} - \mathbf{x}^{w_1^-} \mathbf{y}^{w_2^-} \mathbf{z}^{w_3^-} \rangle$$

that is the defining ideal of the diagonal embedding of $X \hookrightarrow X \times X \times X$. \square

Remark 2.7. Consider the matrix $\mathcal{A} = (0, 1, 2, 3)$ defining the twisted cubic curve in \mathbb{P}^3 . Given the ring $S = k[x_0, \dots, x_3, y_0, \dots, y_3, z_0, \dots, z_3]$, the defining ideal $I_{\mathcal{L}} \neq I_{\Delta}$. Macaulay2 gave six binomials in $I_{\mathcal{L}}$ that are not in I_{Δ} . One such binomial is $x_1^2 y_0 y_3 z_2^2 - x_0 x_2 y_1 y_2 z_1 z_3$.

3 Circuits

The remainder of this paper will discuss the algebro-geometric results related to matrices of the form $\mathcal{A} = (i_1, i_2, i_3, i_4)$. The circuits of the Graver basis play a major role so we begin by stating some well-known results for circuits $\mathcal{C}_{\mathcal{A}}$.

Sturmfels ([21], Lemma 4.9) showed that the 1-norm of a circuit of $\mathcal{A} \subseteq \mathbb{Z}^{d \times n}$ is bounded by the area $\mathcal{D}(\mathcal{A}) = \max\{|\det(a_{i_1}, \dots, a_{i_d})| : 1 \leq i_1 < \dots < i_d \leq n\}$. In particular, there is an upper bound on the 1-norm of a circuit that depends on the rank of the matrix:

Lemma 3.1. *If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ is a circuit of $\mathcal{A} \subseteq \mathbb{Z}^{d \times n}$ then the 1-norm $\|\mathbf{u}\|_1 = \sum_{i=1}^n |u_i| \leq (d+1)\mathcal{D}(\mathcal{A})$.*

Let $0 < a < b$ and denote by $\mathcal{A} = (0, a, b, a+b)$ matrices of the form

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & a+b \end{pmatrix}.$$

This matrix is special because of its geometric properties and the form of its Graver basis. Let $c = a + b$. Then $\mathcal{D}(\mathcal{A}) = \max\{a, b, c, b-a, c-b, c-a\} = c$ and thus by Lemma 1.16, the upper bound on the circuits for this matrix is

Corollary 3.2. *For any circuit $\mathbf{u} \in \ker_{\mathbb{Z}}(\mathcal{A})$, $\|\mathbf{u}\|_1 \leq 3c$.*

The geometric properties of these codim 2 matrices follows by analyzing the Gale diagram associated to the matrix \mathcal{A} .

Definition 3.3. *Given the matrix $\mathcal{A} \subseteq \mathbb{Z}^{2 \times 4}$, let $\mathcal{L}(\mathcal{A}) = \langle \alpha - \beta \in \ker(\mathcal{A}) \rangle$ be the lattice associate to \mathcal{A} and suppose $\mathcal{L}(\mathcal{A}) = \langle a_1, a_2 \rangle$ be the vectors generating the lattice.*

These vectors, written as columns of a new matrix, define a 4×2 integer matrix whose row vectors $b_1, \dots, b_4 \in \mathbb{Z}^2$ define in \mathbb{R}^2 the **Gale diagram** $G_{\mathcal{L}}$ for the ideal $I_{\mathcal{A}}$. The **Gale dual diagram** $G_{\mathcal{L}}^*$ is obtained by rotating the elements in $G_{\mathcal{L}}$ by $\frac{\pi}{2}$ and we denote by the vectors $\{b_1^*, b_2^*, b_3^*, b_4^*\}$, each $b_i^* \in \mathbb{Z}^2$, as the row vectors for $G_{\mathcal{L}}^*$.

The four row vectors in $G_{\mathcal{L}}^*$ correspond to the circuits in the Graver basis $\mathcal{G}r(\mathcal{A})$ for the matrix \mathcal{A} and are extremal rays dividing \mathbb{R}^2 into eight cones. The remaining vectors of $\mathcal{G}r(\mathcal{A})$ are elements in the Hilbert basis of the cones defined by these rays. If b_1^*, b_2^* are two vectors defining a cone $C = \text{cone}(b_1^*, b_2^*)$ and if $\det(b_1 b_2) = 1$, then the cone is unimodular and thus there is exactly one lattice point in the interior of the parallelogram spanned by these two vectors. **Example** The matrix $\mathcal{A} = (0, 1, 3, 14)$. By Macaulay2 [20], the minimal generators for the toric ideal are

$$I_{\mathcal{A}} = \langle (-2, 3, -1, 0), (3, 1, -5, 1) \rangle.$$

Thus the Gale dual diagram is defined by the 4×2 matrix

$$G_{\mathcal{L}}^* = \begin{pmatrix} -3 & -1 & 5 & -1 \\ -2 & 3 & -1 & 0 \end{pmatrix}^T = (b_1^*, b_2^*, b_3^*, b_4^*)^T$$

See Figure (1). The extremal rays defining each 2-dimensional cone correspond to the circuits in the Graver basis and the minimal lattice points in the interior of the cones are Hilbert basis elements. Thus a bound on the circuits $\mathcal{C}(\mathcal{A})$ would be meaningful. Hosten ([12], 2.2.10) conjectured

Conjecture 3.4. *True Circuit Conjecture*

The 1-norm of a Graver basis element $\mathcal{G}r(\mathcal{A})$ is bounded by the 1-norm of a true circuit.

Figure 1: The Gale Dual diagram for $\mathcal{A} = (0, 1, 3, 14)$

Definition 3.5. We denote the maximal 1-norm of the set of circuits of $\mathcal{G}r(\mathcal{A})$ by $\text{maxcircuit}(\mathcal{A})$ and define it as

$$\text{maxcircuit}(\mathcal{A}) = \max\left\{\sum_{i=1}^n |u_i| : \mathbf{u} \text{ is a circuit of } \mathcal{A}\right\}.$$

Assuming the True Circuit Conjecture, the maximal 1-norm of a Graver basis element can be bounded above:

Corollary 3.6. *The maximum degree of any Graver basis element of the toric ideal $I_{\mathcal{A}}$ of a $d \times n$ matrix \mathcal{A} is bounded above by $\text{maxcircuit}(\mathcal{A})$.*

However, for the cases where $\mathcal{A} \subseteq \mathbb{Z}^{2 \times n}$ for $n \geq 5$, it is not true that $\text{maxg}(\mathcal{A}) \leq \text{maxcircuit}(\mathcal{A})$. Naively, the next question to pose is how these bounds are related

to the total degree of any primitive binomial in a homogeneous toric ideal $I_{\mathcal{A}}$. The following Theorem ([21]) gives an upper bound defined by the 1-norm on the elements in the Graver basis defined in terms of the rank of the matrix \mathcal{A} and the number of variables.

Theorem 3.7. *Let $\dim(\mathcal{A}) = d$ and $D(\mathcal{A}) = \max\{| \det(a_{i_1}, \dots, a_{i_d}) | \}$ such that $1 \leq i_1 < i_2 < \dots < i_d \leq n$. The total degree of any primitive binomial in $I_{\mathcal{A}}$ is less than $(d + 1)(n - d)D(\mathcal{A})$.*

One related bound was proved by Hosten [12]

Proposition 3.8. *Let $I_{\mathcal{A}} \subset S$ be an homogeneous toric ideal. Then*

$$\text{reg}(I_{\mathcal{A}}) \leq \frac{n}{2} \max g(\mathcal{A})$$

where $\text{reg}(I_{\mathcal{A}})$ is the Castelnuovo-Mumford regularity for an ideal of a polynomial ring $S = k[x_1, \dots, x_n]$ determined by its minimal free resolution.

The main theorem we will prove is the following:

Theorem 3.9. *Let \mathcal{A} be the homogeneous 2×4 matrix of the form*

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i_1 & i_2 & i_3 & i_4 \end{pmatrix}$$

such that $1 \leq i_1 < i_2 < i_3 < i_4$ and $I_{\mathcal{A}}$ is the defining ideal. Then assuming the True Circuit Conjecture, the Graver complexity is bounded above by the maximum 1-norm in $TC_{\mathcal{A}}$, that is

$$\max\{i_2 + i_3 + i_4 - 3i_1, \quad 3i_4 - i_1 - i_2 - i_3\}.$$

Moreover, if the set of integers $\{i_2 - i_1, i_3 - i_1, i_4 - i_1, i_3 - i_2, i_4 - i_2, i_4 - i_3\}$ are pairwise coprime, then the bound is tight.

4 Algebraic and Geometric Results for matrices

$$\mathcal{A} = (0, a, b, a + b)$$

We would like to distinguish between circuits and true circuits. To do this, we require a way to determine whether two matrices are equivalent.

Lemma 4.1. *If $0 < i_2 < i_3 < i_4$ then*

$$g(\mathcal{A}) = g \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & i_2 & i_3 & i_4 \end{pmatrix} = g \begin{pmatrix} 1 & & 1 & 1 \\ 0 & i_4 - i_3 & i_4 - i_2 & i_4 \end{pmatrix}$$

Proof. Let $\mathcal{A} = \{(1, 0), (1, i_2), (1, i_3), (1, i_4)\}$. Then

$$\ker \mathcal{A} = \ker \begin{pmatrix} 1 & & 1 & 1 \\ -i_4 & i_2 - i_4 & i_3 - i_4 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & & 1 & 1 \\ 0 & i_4 - i_3 & i_4 - i_2 & i_4 \end{pmatrix} = \ker \mathcal{A}'.$$

Thus, the Graver basis for \mathcal{A} is the same as for \mathcal{A}' up to symmetry and hence their Graver representatives are equal. \square

Lemma 4.2. *Row reduction on \mathcal{A} does not alter the Graver basis.*

Proof. $\mathcal{G}r(\mathcal{A})$ depends solely on the $\ker \mathcal{A}$ and the kernel does not change if we do row operations. Moreover, $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ solely depends on $\ker \mathcal{G}r(\mathcal{A})$ which also does not change by row operations on the matrix given by $\mathcal{G}r(\mathcal{A})$. \square

Lemma 4.3. *The circuits of the 2×4 matrix $\mathcal{A} = (0, a, b, a + b)$ are the true circuits.*

Proof. Let $\mathcal{A} = (0, a, b, c)$ with $c = a + b$. By Cramer's determinantal rule ([19]) the 2×2 minors of \mathcal{A} determine the circuits

$$\mathcal{C}(\mathcal{A}) = \{(0, b - c, c - a, a - b), (b - c, 0, c, -b), (a - c, c, 0, -a), (a - b, b, -a, 0)\}.$$

\mathcal{A} is equivalent to itself so by Lemma 4.1, these differences are pairwise coprime.

Therefore the circuits for \mathcal{A} are the true circuits. \square

Corollary 4.4. *For $u \in \mathcal{G}r(\mathcal{A})$, $\text{maxcircuit}(u) = 2(a + b)$.*

Let $d = \text{g.c.d.}(a, b)$. Since $\ker(\mathcal{A}) = \ker(\mathcal{A}_{\{a/d, b/d\}})$ we have

Lemma 4.5. $g(\mathcal{A}) = g(\mathcal{A}_{\{a/d, b/d\}})$.

Therefore, we may assume that $\text{g.c.d.}(a, b) = 1$.

Lemma 4.6. *The kernel of the matrix $\mathcal{A} = (0, a, b, a + b)$ is generated by*

$$\lambda = (1, -1, -1, 1) \text{ and } g_0 = (b, -(a + b), 0, a).$$

Proof. Write the matrix \mathcal{A} in its Hermite normal form $\mathcal{A} = T \cdot U$, where $T = (B \mid 0)$, B is lower triangular, and U is the unimodular matrix defining the integer column operations. After a column operation, we get the matrix U

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the last column is a minimal generator of the kernel of \mathcal{A} . Let $\lambda = (1, -1, -1, 1)$. This U corresponds to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \end{pmatrix}$. Notice that the first three coordinates in the last row of U is in the kernel of a restricted matrix $(0, a, b)$. Other elements in the kernel of $(0, a, b)$ will have the form $ay+bz = 0$ which implies that a unique solution is $(-b, a)$. Therefore, $(b - a, -b, a)$ is the unique solution to the restricted matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & a & b \end{pmatrix}$. Now λ can be used to transform any vector $v \in \ker \mathcal{A}$ into $v + \alpha\lambda$ with zero as the last component. In particular, take $(b - a, -b, a, 0) + (a, -a, -a, a) = (b, -(a + b), 0, a)$ which is a unique minimal generator for the kernel of \mathcal{A} . \square

Lemma 4.7. For $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & a + b \end{pmatrix}$,

$$\mathcal{G}r(\mathcal{A}) = \{g_0, g_0 + \lambda, g_0 + 2\lambda, \dots, g_0 + (a + b)\lambda, \lambda\},$$

where $g_0 = (b, -(a + b), 0, a)$

and $\lambda = (-1, 1, 1, -1)$.

Proof. The lattice associated to the kernel of \mathcal{A} is generated by

$g_0 = (b, -(a + b), 0, a)$ and $\lambda = (-1, 1, 1, -1)$. Notice that the set $\mathcal{G}r(\mathcal{A})$ in the hypothesis generates $\ker_{\mathbb{Z}}(\mathcal{A})$ because it contains λ, g_0 . Using Algorithm 2.7.1 in Hemmecke [10] (see also Pottier [17]), we must show this set is a minimal set of irreducible vectors. Notice that $g_0 \sqsubseteq g_0 - \lambda$ and that $g_0 + (a + b)\lambda \sqsubseteq (g_0 - (a + b)\lambda) + \lambda$. The difference of any two elements in the set $\mathcal{G}r(\mathcal{A})$ is some multiple of λ so there are

no new elements obtained from taking differences of the sets. It remains to consider possible sums of elements in the generating set and determine whether or not these sums are irreducible. Consider the sum of any two elements in the set $\mathcal{Gr}(\mathcal{A})$, say $g_0 + p\lambda + g_0 + q\lambda = 2g_0 + (p + q)\lambda$. If we find an element $g_0 + r\lambda \sqsubseteq 2g_0 + (p + q)\lambda$ then $2g_0 + (p + q)\lambda$ is reducible.

There is a structure to the set $\mathcal{Gr}(\mathcal{A})$. The generators λ, g_0 actually define a line of negative slope. See Figure (2). None of the elements on this line are divisible by the other elements and therefore they are minimal. If $2 \mid p + q$ then take $r = \frac{p+q}{2}$ and hence $g_0 + r\lambda = 2g_0 + (p + q)\lambda \sqsubseteq 2g_0 + (p + q)\lambda$. If 2 does not divide $p + q$, take $r = \frac{p+q\pm 1}{2}$. Then $g_0 + r\lambda = 2g_0 + (p + q \pm 1)\lambda$ which is an even multiple of λ and we use the first case. Thus we have found an r dividing the sum of any two elements in $\mathcal{Gr}(\mathcal{A})$ so that element is not minimal. \square

Let B_c be the matrix

$$B_c = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 2 & \cdots & c & 1 \end{pmatrix}.$$

Theorem 4.8. *The Graver complexity $g(\mathcal{A})$ of $\mathcal{A} = (0, a, b, a + b)$ is equal to the maximal 1-norm of a Graver basis element of B_c , where $c = a + b$.*

Proof. By Lemma 4.7,

$$\mathcal{Gr}(\mathcal{A}) = \{g_0, g_0 + \lambda, g_0 + 2\lambda, \dots, g_0 + (a + b)\lambda, \lambda\},$$

where $g_0 = (b, -(a + b), 0, a)$ and $\lambda = (-1, 1, 1, -1)$.

Figure 2: Graver basis of $\mathcal{A} = (0, a, b, a + b)$

Now let $c = (a + b)$ and let $(\alpha_0, \alpha_1, \dots, \alpha_c, \beta) \in \ker(\mathcal{G}r(\mathcal{A}))$ which is the case if and only if

$$\left(\sum_{i=0}^c \alpha_i \right) g_0 + \left(\sum_{i=0}^c i\alpha_i + \beta \right) \lambda = 0. \quad (3)$$

Since g_0 and λ are linearly independent, this implies that the coefficients in (3) must vanish. Therefore, the kernel of $\mathcal{G}r(\mathcal{A})$ is the same as the kernel of the matrix

$$B_c = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 2 & \cdots & c & 1 \end{pmatrix}$$

implying $\mathcal{G}r(\mathcal{G}r(\mathcal{A})) = \mathcal{G}r(B_c)$, and the result follows. \square

The endpoints of the line defined by the two generators λ and g_0 of the Graver basis are always circuits.

Corollary 4.9. *The cardinality of the Graver basis of the matrix \mathcal{A} is $c + 2 = (a + b) + 2$.*

Since g_0 is defined in terms of a, b and every other element in both the Graver basis for \mathcal{A} , the Graver complexity of $\mathcal{G}r(\mathcal{A})$ will be a number defined in terms of $(a + b)/d$, where $d = \gcd(a, b)$. Thus

Corollary 4.10. *The Graver complexity $g(\mathcal{A})$ is a function in $\frac{(a+b)}{d}$.*

As a result of the relationship between $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ and $\mathcal{G}r(B_{a+b})$, we would like to be able to quickly find the maximal 1-norm for elements in the Graver basis of B_c , for arbitrary c . This leads us to the following conjecture:

Conjecture 4.11. *B_c Conjecture*

For $c \geq 2$, the maximal 1-norm of elements in $\mathcal{G}r(B_c)$ is $2c$.

This conjecture is substantiated by examples in 4Ti2 for $3 \leq c \leq 18$.

A general condition for the Graver representative in $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ in terms of circuits is given by

Proposition 4.12. *A Graver representative $g \in \mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ for a matrix \mathcal{A} has nonzero coordinates corresponding to at least two circuits in $\mathcal{G}r(\mathcal{A})$.*

Proof. The graver representative $g \in \mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ represents a linear combination of the elements in the Graver basis; the coordinates in g are the "worst" or largest coefficients of the elements $\{g_1, \dots, g_k\} = \mathcal{G}r(\mathcal{A})$.

Since the support of a circuit of $\mathcal{G}r(\mathcal{A})$ is 3, consider a general element $(a_0, a_1, a_1, \dots, a_c, b) \in \mathcal{G}r(B_c) = \mathcal{G}r(\mathcal{G}r(\mathcal{A}))$. If $a_0 = 0$ then $(a_1, \dots, a_c, b) \in \mathcal{G}r(B_{c-1})$, or if $a_c = 0$ then $(a_0, \dots, a_{c-1}, b) \in \mathcal{G}r(B_{c-1})$ and the claim follows by induction. Thus consider the case where $a_0 \neq 0, a_c \neq 0$. But $a_0 \neq 0, a_c \neq 0$ corresponds to a vector in $\mathcal{G}r(B_c)$ that has nonzero coordinates as the coefficients of the column vectors $(1, 0)^T, (1, c)^T$. These are precisely the two endpoints of the line, $g_0, g_0 + (a + b)\lambda$ and are circuits. □

5 Coverings By Circuits

To better understand the geometry associated to the Graver basis, we need a definition that describes the relationship between the circuits and the noncircuits of \mathcal{A} .

Definition 5.1. *The Graver basis for an integer $d \times n$ matrix \mathcal{A} is **covered by circuits** if, in each orthant, the Hilbert basis of that orthant lies in the simplex spanned by the circuits defining that orthant.*

For example, consider Figure (1) which is the Gale dual diagram of the matrix $(0, 1, 3, 14)$. Within each cone, the extremal rays have largest 1-norm. Matrices of the form $(0, a, b, a + b)$ have a unique form of Gale dual diagram. Consider the matrix $(0, 1, 2, 3)$ in Figure (3) representing the twisted cubic curve in \mathbb{P}^3 . Its Gale dual diagram is defined by the matrix $G_{\mathcal{L}}^* = \begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 1 & -1 \end{pmatrix}^T$.

Figure 3: Gale Dual Diagram for $(0, 1, 2, 3)$

We have the following useful result:

Theorem 5.2. *The Graver basis $\mathcal{G}r(\mathcal{A})$ of the matrix $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & a+b \end{pmatrix}$ is covered by circuits.*

Proof. By Lemma 4.3, the maximal 1-norm for elements in the Graver basis is $2(a+b)$ and is taken on by a circuit. \square

Theorem 5.3. *Given the matrix $\mathcal{A} = (0, a, b, a+b)$ the region bounded by the circuits in \mathbb{R}^2 has area $2c$.*

Proof. Construct $G_{\mathcal{L}}^*$. From Theorem 5.2, for each cone there is at least one of the vectors defining the extremal rays that has norm greater than or equal to any other element restricted to that cone. Denote by \mathcal{P} the region formed from connecting the lattice points corresponding to the circuits of $\mathcal{G}r(\mathcal{A})$. Since these vectors of $G_{\mathcal{L}}^*$ define a central hyperplane arrangement, there is symmetry about the origin. Hence \mathcal{P} is a parallelogram and it is defined by those vectors in $G_{\mathcal{L}}^*$ with the largest norms. We claim the area of the parallelogram is $2c$.

From above, the vector $\lambda = (1, -1, -1, 1)$ lies in the kernel of \mathcal{A} and we may take any other vector in $\mathcal{G}r(\mathcal{A})$ that is nonconformal to λ as the other vector defining $G_{\mathcal{L}}^*$. The Gale diagram is determined by two vectors of minimal length. Using the λ, g_0 previously determined, choose λ and the vector $v = (b, -c, 0, a) + 2\lambda = (b-2, -c+2, 2, a-2)$ to define $G_{\mathcal{L}}$. From the set of vectors in $G_{\mathcal{L}}^*$, the two of largest norm are

$\{\pm(-1, c - 2), \pm(-1, -2)\}$ and these define the parallelogram determining the area $2 \cdot (c - 2 + 2) = 2c$. \square

For arbitrary matrices $\mathcal{A} = \{(1, i_1), (1, i_2), (1, i_3), (1, i_4)\}$ with $0 \leq i_1 < i_2 < i_3 < i_4$ its Graver basis is also covered by circuits but the Gale dual diagram does not have the same nice symmetry. The proof, due to Oda [16], uses plane convex geometric properties that fail for dimensions greater than 2.

Let $N \cong \mathbb{Z}^2$ and let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ with $\dim(\sigma) = 2$. Choose vectors $n, n' \in \mathbb{Z}^2$ defining the rays of the cone σ such that they are **minimal** or **primitive** in the sense that they are not positive integer multiples of any elements of N except themselves and they are \mathbb{R} -linearly independent. Thus $\sigma = \text{cone}(n, n')$. The lattice points in each cone arising as a result of doing continued fractions define the structure for that cone.

Definition 5.4. *Let σ be a cone in \mathbb{R}^2 defined as $\sigma = \text{cone}(c_1, c_2)$. A **corner** in a cone is an interior Hilbert basis element h whose 1-norm satisfies*

$$\|h\|_1 \leq \min\{\|c_1\|_1, \|c_2\|_1\}.$$

In Figure 4, the lattice points l_1, l_2, l_3 are the only corners in the cone σ . All other lattice points lie on two lines. The matrix $(0, , 1, 3, 14)$ in Figure 1 is an example where there are two corners in a cone. The following theorem is due to Oda ([16]). We give a proof because it is descriptive.

Theorem 5.5. *For a 2-dimensional cone $\sigma = \text{cone}(n, n')$, let Θ be the convex hull in $\mathbb{N}_{\mathbb{R}}$ $\text{conv}(\sigma \cap \mathbb{Z}^2) \setminus \{O\}$. Let $l_0 = n, l_1, \dots, l_s, l_{s+1} = n'$ in this order be the points of*

Figure 4: The boundary polygon $\partial\Theta$ of the cone σ

\mathbb{Z}^2 lying on the compact edges of the boundary polygon $\partial\Theta$. Let Δ' be the subdivision of σ obtained as the set of faces of $\text{cone}(l_{j-1}, l_j)$ such that $1 \leq j \leq s+1$. Then the vertices of the convex polygon in each 2-dimensional cone in \mathbb{R}^2 have minimal length with respect to that cone.

Proof. The boundary polygon $\partial\Theta$ is the union of two half lines starting from n and n' and the line segments joining l_{j-1} and l_j , for $1 \leq j \leq s+1$. By the choice of the l_0, l_1, \dots, l_{s+1} , the triangle with vertices at the origin O, l_{j-1} and l_j contains no points in \mathbb{Z}^2 other than the vertices. Thus $\{l_{j-1}, l_j\}$ is necessarily an integer basis of \mathbb{Z}^2 because of plane convex geometry. Hence, Δ' is a nonsingular fan and is the coarsest subdivision of σ . Since l_{j-1} and l_{j+1} lie on mutually opposite sides with respect to l_j , and since $\{l_{j-1}, l_j\}$ and $\{l_j, l_{j+1}\}$ are both integer bases of \mathbb{Z}^2 , there exists $a_j \in \mathbb{Z}$ such that $l_{j-1} + l_{j+1} + a_j l_j = O$. By the convexity of Θ , $a_j \leq -2$. This construction

of Δ' requires an algorithm to compute the vertices of the convex hull Θ .

Since $n, n' \in \mathbb{Z}^2$ are \mathbb{R} -linearly independent primitive elements in σ , we can find a primitive element $n_1 \in \mathbb{Z}^2$ and relatively prime integers p, q with $0 \leq p < q$ such that $\{n, n_1\}$ is a \mathbb{Z} -basis of \mathbb{Z}^2 and that $n' = pn + qn_1$. Note that $q = 1$ implies that $p = 0$ which implies that σ is nonsingular.

Let $b_j = -a_j$, for $1 \leq j \leq s$. Define subsets $\{l_0, l_1, \dots, l_{s+1}\}$ and $\{n_2, \dots, n_{s+1}\}$ of \mathbb{Z}^2 inductively by

$$l_0 := n, \quad l_j := l_{j-1} \quad \text{and} \quad n_{j+1} := (b_j - 2)l_{j-1} + (b_j - 1)n_j,$$

where $l_0, l_1, \dots, l_{s+1} = n'$ in this order are points lying on the compact edge of the boundary polygon $\partial\Theta$ and $l_{j-1} + l_{j+1} = b_j l_j$. \square

Therefore, doing finite continued fractions guarantees that the vertices along the edge of a convex polygon in a cone in \mathbb{R}^2 will be of minimal length and are, therefore, Hilbert basis elements for that cone.

Theorem 5.6. *The Graver basis $\mathcal{G}r(\mathcal{A})$ is covered by circuits.*

Proof. Let $\{b_1^*, \dots, b_4^*\} \in G_{\mathcal{L}}^*$ be the row vectors defining the Gale dual diagram. By Theorem 5.5, every two consecutive vectors in $G_{\mathcal{L}}^*$ define $\text{cone}(b_i^*, b_{i+1}^*)$ whose interior Hilbert basis elements h lie on the boundary polygon. Therefore $\|h\|_1 \leq \max\{\|b_i^*\|_1, \|b_{i+1}^*\|_1\}$. Since this is true for every i , the result follows. \square

Unlike the Graver basis for $\mathcal{A} = (0, a, b, a + b)$, the circuits of B_c for arbitrary $c \geq 3$ are not the same as the true circuits.

Lemma 5.7. *The circuits of a $2 \times (c+2)$ integer matrix B_c are not the true circuits for B_c .*

Proof. By Cramer's rule, the possible 2×2 minors define the true circuits $\mathcal{TC}(B_c)$.

Given $B_c = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & \cdots & c-1 & c & 1 \end{pmatrix}$ Thus there are three cases of circuits to consider.

Case 1: The true circuit is defined by any three consecutive nonzero coordinates

$$(0, \dots, \begin{vmatrix} 1 & 1 \\ s & s+1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ s-1 & s+1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ s-1 & s \end{vmatrix}, 0, \dots)$$

which is the vector $(0, \dots, 1, 2, 1, 0, \dots)$ and thus the g.c.d. of the coordinates is one.

Case 2: The true circuit is defined by any two consecutive nonzero coordinates and one separate $(0, \dots, s, s+1, 0, \dots, k, \dots, 0)$

$$(0, \dots, \begin{vmatrix} 1 & 1 \\ s+1 & k \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ s & k \end{vmatrix}, 0, \dots, \begin{vmatrix} 1 & 1 \\ s & s+1 \end{vmatrix}, \dots, 0)$$

which is the vector $(0, \dots, k-s-1, k-s, 0, \dots, 1, \dots, 0)$ and the g.c.d. of the components is one.

Case 3: All three nonzero coordinates are separate $(0, \dots, i_s, \dots, 0, i_t, 0, \dots, i_k, 0, \dots)$

where $s < t < k$.

$$(0, \dots, \begin{vmatrix} 1 & 1 \\ t & k \end{vmatrix}, \dots, 0, \begin{vmatrix} 1 & 1 \\ s & k \end{vmatrix}, 0, \dots, \begin{vmatrix} 1 & 1 \\ s & t \end{vmatrix}, 0, \dots)$$

This is the vector $(0, \dots, k-t, \dots, 0, k-s, 0, \dots, t-s, 0, \dots)$. If the s, t, k are all even, then the g.c.d. of the components is not equal to one. However, when the s, t, k are all odd, we get a g.c.d. equal to one or more. Thus having s, t, k all odd is not sufficient to have the g.c.d. of the coordinates equal to one. For example, in B_{11} , $\begin{vmatrix} 1 & 1 \\ 11 & 9 \end{vmatrix} = 2$, $\begin{vmatrix} 1 & 1 \\ 11 & 3 \end{vmatrix} = 8$, $\begin{vmatrix} 1 & 1 \\ 9 & 3 \end{vmatrix} = 6$. The g.c.d. is 2. Therefore, the true circuits are not equal to the circuits. \square

Remark 5.8. *Given a matrix $\mathcal{A} \subseteq \mathbb{Z}^{2 \times n}$, then there are $n - 2$ generating vectors for the lattice $\mathcal{L}(\mathcal{A})$ and therefore $\binom{n}{n-3}$ true circuits in $\mathcal{G}r(\mathcal{A})$. For B_c , there are $\binom{c+2}{c-1} = \frac{1}{6}(c^3 + 3c^2 + 2c)$ circuits.*

Consider B_3 and look at the orthant defined by the following Graver basis elements $\mathcal{G}r(B_3)$:

$(1, -2, 1, 0, 0)$, $(1, -2, 0, 1, -1)$, $(0, -1, 1, 0, -1)$, $(0, -1, 0, 1, -1)$. Write the non-circuit as a linear combination of the three circuits:

$$h = (1, -2, 0, 1, -1) = 1(1, -2, 0, 1, -1) + 1(0, 1, -1, 0, 1) + 1(0, -1, 0, 1, -1)$$

where the height of the Hilbert basis element in that orthant is $\text{height}(h) = 3 > 1$. Thus the element h lies outside the simplex defined by the circuits and therefore $\mathcal{G}r(B_3)$ is not covered by circuits. In general, many examples can be found where interior Hilbert basis vectors are not covered by the circuits. Thus we have

Claim 5.9. *The Graver basis of B_c , $c \geq 3$ is not covered by circuits.*

Since the Graver of the Graver basis of the matrix $\mathcal{A} = (0, a, b, a + b)$ has kernel isomorphic to the kernel of B_{a+b} , we have

Corollary 5.10. *The Graver basis of the Graver basis of $\mathcal{A} = (0, a, b, a + b)$ is not covered by circuits.*

Therefore, we are forced to assume the True Circuit Conjecture to prove the Theorem 8.6.

6 Structure and Generating Functions of codim 2 Graver Bases

In this section we deal with the following problem: Given a $(n - 2) \times n$ matrix A of codim 2, compute the Graver basis $Gr(A)$ of A . As we will see, Graver bases of codim 2 matrices have a very nice and simple structure: they are the integer points on a collection of line-segments. Clearly, once we know these line-segments, we can easily write down the generating function corresponding to $Gr(A)$, a polynomial-size encoding of the Graver basis proven to exist by Barvinok and Woods [1].

In the following, we will present a polynomial-time algorithm (in the bit-length of the maximal entry in A), that computes these line-segments.

The algorithm consists of the following major steps:

1. Compute the n circuits (and their negatives) of A .
2. Form the Gale dual diagram (in dimension 2). The circuits divide the plane into $2n$ simplicial cones. The Graver basis elements of A are in one-to-one correspondence with the Hilbert bases elements in all of those cones.
3. Compute the Hilbert basis for each simplicial cone: Find the endpoints of the line segments of the Hilbert basis recursively, one by one.
4. Lift the 2-dimensional Graver basis vectors back to the original n -dimensional space.
5. Use the line-segments to write down the generating function of $Gr(A)$.

6.1 Computing the Hilbert Basis of a 2-dimensional Cone

The main structural fact of 2-dimensional Hilbert bases that we will employ is the following restatement of the Theorem 5.5:

Theorem 6.1. *The Hilbert basis of the 2-dimensional cone generated by the vectors (a_1, a_2) and (c_1, c_2) consists exactly of all integer points on the convex hull of all non-zero integer points. In other words, the Hilbert basis elements lie on certain line-segments, whose number is bounded by a polynomial in the bit length of $a_1, a_2, c_1,$ and c_2 .*

Thus, instead of computing the potentially exponential-size Hilbert basis, we compute only the end points of the polynomially many line-segments together with the line directions. This gives a polynomial-size encoding of all Hilbert basis elements.

Proposition 6.2. *Let $\mathcal{A} = (0, i_2, i_3, i_4)$ such that $0 < i_2 < i_3, i_4$ be integers. Let (c_1, c_2) be an extremal ray in \mathbb{Z}^2 and suppose that $\{(c_1, c_2), (u_1, u_2)\}$ span \mathbb{Z}^2 . Then for*

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = p \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

q is uniquely determined and $q = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Proof. Following Theorem 5.5, $c = (c_1, c_2)$ and $a = (a_1, a_2)$ are extremal rays in \mathbb{Z}^2 corresponding to circuits in the Graver basis of \mathcal{A} . The system of equations in the

proposition implies that $\begin{vmatrix} u_1 & c_1 \\ u_1 & c_2 \end{vmatrix} = 1$. Since $c = (c_1, c_2)$ is a circuit, the
g.c.d. $(c_1, c_2) = 1$. Thus $\begin{vmatrix} u_1 & c_1 \\ u_1 & c_2 \end{vmatrix} = \begin{vmatrix} \frac{1}{q}(a_1 - pc_1) & c_1 \\ \frac{1}{q}(a_2 - pc_2) & c_2 \end{vmatrix} = -1$ which implies

$$\frac{1}{q}c_2(a_1 - pc_1) - \frac{1}{q}c_1(a_2 - pc_2) = -1.$$

Thus $a_1c_2 - a_2c_1 = -q$, i.e. $q = a_2c_1 - a_1c_2$ and q is uniquely determined. \square

There is not, however, a method for finding an unique integer p .

Proposition 6.3. *To find $p \in \mathbb{Z}$ satisfying Proposition 6.2 solve one of the two equivalences:*

$$a_1 - pc_1 \equiv 0 \pmod{q} \quad a_2 - pc_2 \equiv 0 \pmod{q}.$$

Proof. These two equivalences above always exist and arise as a result of writing an interior lattice point as an integer combination of the extremal rays. Use the equivalence where the $c_i \neq 0$. Suppose $c_1 \neq 0$ and find p from $a_1 - pc_1 \equiv 0 \pmod{q}$ where $0 < p < q$ and $(p, q) = 1$. There exists r such that $(a_1 - pc_1)r = q \Rightarrow pr = \frac{a_1 - q}{c_1}$. To get the congruences, note that q needs to divide the g.c.d. $(a_1 - pc_1, a_2 - pc_2)$ which implies that

$$\alpha(a_1 - pc_1) + \beta(a_2 - pc_2) = \gamma \cdot q.$$

The "mod q " statement says that given q and c_1 , there exists a unique $c_1^{-1} \pmod{q}$.

Thus find $c_1^{-1} \in [0, q-1]$ with $c_1 c_1^{-1} \equiv 1 \pmod{q}$. If $p = a_2 c_2^{-1} = a_1 c_1^{-1} \pmod{q}$ then $a_2 c_2^{-1} - a_1 c_1^{-1} \equiv \frac{a_2 c_1 - a_1 c_2}{c_1 c_2} \pmod{q} = 0$. \square

We give the following example to demonstrate the methodology behind using continued fractions for finding the corners and interior Hilbert basis elements for codim 2 matrices. **Example** Let $\mathcal{A} = (0, 3, 4, 5)$ and let $C = \text{cone}(c_1, c_2)$ where

$$c_1 = (-1, 2), \quad c_2 = (-2, 1), \quad q = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3.$$

Consider $0 < p < 3$. If $p = 2$, then $u = (0, -1)$ and $c_1 + u = (-1, 1) \in \text{int}(C \cap \mathbb{Z}^2)$. Using c_2 instead, $q = \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} = 5$ so that $0 < p < 5$. In the equation $(2, 1) = p(-1, 2) + 5(u_1, u_2)$, if $p = 3$ then $u = (1, -1)$. Thus $(-1, 2) + (1, -1) = (0, 1)$ so that $c_1 + 2u = (1, 0) \notin \text{int}(C)$ which implies that $(0, 1)$ is a corner in the cone C .

To iterate the process, let $c_3 = (0, 1)$ and find u for the cone defined by c_2, c_3 . Here $q = 2$, $0 < p < 2$ and thus

$$(2, 1) = p(0, 1) + q(u_1, u_2) \Rightarrow u = (1, 0)$$

simplifies to $c_3 = (0, 1) + u = (1, 1) \notin \text{int}(C)$. Moreover, $c_3 \neq c_2$ and $c_3 + 2u = (0, 1) + (2, 0) = (2, 1) = c_2$ and the process is complete. Thus the elements in the interior of the cone are precisely the Hilbert basis elements for that cone.

6.2 Constructing the Generating Function of $Gr(A)$

We want to relate the above process to the results of Barvinok ([1]).

The generating function

$$g_{Gr(A)}(z) = \sum_{\alpha \in Gr(A)} z^\alpha$$

of $Gr(A)$ can now be written as

$$\begin{aligned} g_{Gr(A)}(z) &= \sum_{\text{line segments } L_i \text{ of } Gr(A)} g_{L_i}(z) - \sum_{\text{end points } e \text{ of line segments}} z^e \\ &= \sum_{L_i} \sum_{j=0}^{k_i} z^{a_i + j\lambda_i} - \sum_e z^e \\ &= \sum_{L_i} a_i \left(\frac{z^{(k_i+1)\lambda} - 1}{z^\lambda - 1} \right) - \sum_e z^e \end{aligned}$$

To implement the notions of Barvinok for the Graver basis once we have the Graver basis elements, we utilize the structure of the basis to write the generating function.

Example Consider the matrix $\mathcal{A} = (0, 1, 2, 3)$ associated to the twisted cubic curve. *4ti2* gives Graver basis elements:

$$\{(-1, 2, -1, 0), (-2, 3, 0, -1), (-1, 1, 1, -1), (0, 1, -2, 1), (-1, 0, 3, -2)\}$$

For ease of notation, let $g_0 = (-1, 0, 3, -2)$ and $L = (1, -1, -1, 1)$.

The 10 vectors in $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ are

$$\{g_0, g_0 + L, g_0 + 2L, g_0 + 3L, L, 0, -L, -g_0, -g_0 - L, -g_0 - 2L, -g_0 - 3L\}$$

where we kill zero in the end. These define three line segments. Therefore, we get the generating function:

$$\begin{aligned}
g_{G(A)}(z) &= z^{g_0} + \dots + z^{g_0+3L} + z^L + z^0 + z^{-L} + \\
&\quad + z^{-g_0} + \dots + z^{-g_0-3L} - z^0 \\
&= z^{g_0} \cdot (1 + z^L + z^{2L} + z^{3L}) + z^{-L} \cdot (1 + z^L + z^{2L}) + \\
&\quad + z^{-g_0} \cdot (1 + z^{-L} + z^{-2L} + z^{-3L}) - 1 \\
&= z^{g_0} \cdot (1 - z^{4L}) / (1 - z^L) + z^{-L} \cdot (1 - z^{3L}) / (1 - z^L) + \\
&\quad + z^{-g_0} \cdot (1 - z^{-L^4}) / (1 - z^{-L}) - 1
\end{aligned}$$

Substitute in both g_0 and L . The result will always have this type of description; only the exponents and g_0 will vary for different examples. Therefore, we have a short representation of a generating function in terms of rational functions.

7 $2c$ Conjecture for B_c

There are a number of ways to approach finding the maximal 1-norm for circuits of the matrix B_c . We may consider its associated semigroup or use primitive partition identities. Proving the B_c Conjecture would demonstrate that the maximum 1-norm for the general case of a $2 \times n$ matrix generalizes the idea that the maximal 1-norm is taken on by a true circuit.

$$\text{Let } B_c = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 2 & 3 & \cdots & c & 1 \end{pmatrix}.$$

Remark 7.1. *The matrix B_1 represents the degenerate case. The kernel of B_1 is equal to its Graver basis $\mathcal{Gr}(B_1) = \{(1, -1, 1)\}$. Thus the maximal 1-norm in $\mathcal{Gr}(B_1)$ is 3 and it is given by a vector with full support.*

Example Consider B_2 . The kernel of B_2 is given by two circuits $\{(0, 1, -1, 1), (-1, 1, 0, -1)\}$ and they generate a lattice. Thus the Gale dual diagram associated to this lattice is given by the four vectors $\{(1, 0), (-1, 1), (0, -1), (1, 1)\}$ which define unimodular cones. Therefore, every element in $\mathcal{Gr}(B_2)$ is a circuit and thus the Graver basis is covered by circuits.

Generalizing this to the case where $c \geq 3$ poses a problem geometrically since for a $2 \times n$ integer matrix \mathcal{A} , there are $n - 2$ elements generating the lattice $\mathcal{L}(\mathcal{A})$.

Lemma 7.2. *Given the matrix B_3 , the Graver basis is defined by*

$$\mathcal{L}(B_c) = \{g_0 = (2, -3, 0, 1, 0), \lambda = (-1, 1, 1, -1, 0), \mu = (-1, 1, 0, 0, -1)\}.$$

Proof. Using 4ti2 given the generators

$$\begin{aligned} \mathcal{Gr}(B_3) = & \{g_0, g_0 + \lambda, g_0 + 2\lambda, g_0 + 3\lambda, g_0 + \mu, g_0 + 2\mu, g_0 + 3\mu, \\ & g_0 + \lambda + \mu, g_0 + 2\lambda + \mu, g_0 + \lambda + 2\mu, \mu - \lambda, \lambda, \mu\}. \end{aligned}$$

□

By the explicit generating set for B_3 , $\|g_0\|_1 = \|g_0 + 3\lambda\|_1 = 6$ where both $g_0, g_0 + 3\lambda$ are circuits. Therefore, for $c = 3$ the $2c$ -conjecture is satisfied:

Corollary 7.3. *The Graver basis for B_3 has maximal L_1 -norm 6.*

Let $S = \mathbb{N} \cdot B_c$ be the semigroup defined by B_c . The semigroup ring associated to B_c is the Noetherian ring $R = \mathbb{Z}[S]$ where the grading is given by the columns of B_c . Let

$$\mathbb{R}_+(S) = \left\{ \sum_{i=1}^n a_i v_i : a_i \in \mathbb{R}_+, v_i \in S, n \in \mathbb{N} \right\}$$

be the cone generated by the semigroup S . For an affine semigroup S let $\text{relint}(S) = S \cap \text{relint}(\mathbb{R}_+ S)$. We use the following from [4] (Lemma 6.1.6):

Lemma 7.4. *Let S be a positive affine semigroup. Then for $c \in \text{relint}(S)$, the ideal generated by the elements x^c is a radical ideal and is contained in every nonzero graded radical ideal of $k[S]$.*

Any Graver basis element is a nonzero vector $u \in \mathbb{Z}^n$ that satisfies $\sum u_i a_i = 0$, where a_i are the columns of the matrix \mathcal{A} . By Lemma 7.4, the toric ideal generated by the elements x^u , where u lies in the relative interior of the positive semigroup, is a radical ideal. The exponent vectors u in the $\text{relint}(S)$ are the Graver basis

elements associated to the matrix B_c . Thus B_c has associated semigroup ring $S = \mathbb{Z}[s, st, st^2, \dots, st^c, t] \subset \mathbb{Z}^2$. The semigroup S defines a pointed cone and therefore we may perform a $SL_2(\mathbb{Z})$ transformation to place the $c + 2$ columns of B_c into the first orthant. Thus the polytope defined by the column vectors of B_c lies in an affine hyperplane.

We may use semigroups to determine which vectors are circuits in B_c .

Proposition 7.5. *Given B_c , $u = (u_0, u_1, \dots, u_{c+1}) \in \mathcal{G}r(B_c)$ with $u_{c+1} > 0$, then u is a circuit.*

Proof. Suppose u is not a circuit that, in particular, satisfies $\text{supp}(u) > 3$, where 3 is the $\text{rank}(B_3) + 1$. For some i , if $u_i > 0$ then $u_{i+1} \geq 0$, and for every $j < i$ and $i < j < c + 1$ we have $u_j \leq 0$. Thus the semigroup of \mathbb{Z}^2 may be divided into two cones,

$$C_1 = \text{Pos}\left\{\begin{pmatrix} 1 \\ j \end{pmatrix} \mid j \geq i, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \quad C_2 = \text{Pos}\left\{\begin{pmatrix} 1 \\ j \end{pmatrix} \mid j < i\right\}$$

defined by the column vectors of the matrix B_c .

Consider the intersection $C_1 \cap C_2$ of the two cones, Figure 5.

By Lemma 7.4, vectors in the Graver basis of B_c lie in the relative interior of the cone defined by the semigroup $\mathbb{N}B_c$. Since the intersection $C_1 \cap C_2 = \emptyset$, any vector $u \in \mathcal{G}r(B_c)$ joining cone C_1 with C_2 will not have zero coordinate sum. In particular, there is no element in $\mathcal{G}r(B_c)$ that connects the point $(0, 1)$ in the semigroup to any other point in the cone C_1 . Therefore, u is a circuit. \square

Figure 5: The Intersection of the cones C_1 and C_2 in the semigroup of B_c .

7.1 Primitive Partition Identities

We recall some definitions that will be necessary for proving some things related to B_c , where $c = a + b$.

Definition 7.6. Fix $c > 0$. Then for $0 < a_i < b_j \leq c$, a **partition identity** of degree $k + l$ is

$$a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_l \quad (4)$$

A *partition identity* is a **primitive** if there is no proper subidentity

$$a_{i_1} + \cdots + a_{i_r} = b_{j_1} + \cdots + b_{j_s} \quad (5)$$

with $1 \leq r + s \leq k + l - 1$. A *partition identity* is an **homogeneous primitive**

partition identity if $k = l$ and there is no proper subidentity with $r = s$. The **largest part** of a partition identity is the largest integer of the a_i, b_j .

Consider the homogeneous matrix

$$H_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

that defines a monomial curve. The following theorem gives an upper bound on the degree of the Graver basis elements for an homogeneous matrix H_n :

Theorem 7.7. *Sturmfels*

An homogeneous matrix of the form H_n has maximum 1-norm equal to $2(n - 1)$ and there are exactly $\phi(n - 1)$ maximal solutions.

Thus given the matrix H_{c+1} , which corresponds to the first c columns of the matrix B_c , any h.p.p.i. 7.6 with $k = l$ will satisfy $2k \leq 2c$. We want to show that the matrix B_c also has the degree bound of $2c$:

$$B'_c = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 2 & 3 & \cdots & c+1 & 1 \end{pmatrix} = \left(H_{c+1} \left| \begin{array}{c} 0 \\ 1 \end{array} \right. \right).$$

Remark 7.8. *Homogeneous primitive partition identities correspond to the Graver basis elements $x_{a_1} \cdots x_{a_k} - x_{b_1} \cdots x_{b_k}$ for the matrix H_n .*

The system $H_c \cdot x = 0$ is an **homogeneous system** and

$$B_c \cdot x = 0 \quad \text{is equivalent to} \quad H_c \cdot x = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

is an inhomogeneous system, written in terms of the matrix H_c for which we have known results, $b > 0$. Therefore, finding the cardinality of the number of elements on

either side of the h.p.p.i. is equivalent to obtaining a degree bound on the Graver basis elements for B_c and this, in turn, is equivalent to finding the minimal solutions to the system of equations. Solving any of these equivalent problems, will prove the $2c$ conjecture:

Conjecture 7.9. *2c Conjecture*

The maximal 1-norm of any element in the Graver basis of the matrix B_c is $2c$, where $c \geq 3$.

We continue now with some more structural results for B_c .

Remark 7.10. *Circuits of B_c of minimal 1-norm are vectors in $u \in \mathcal{G}r(B_c)$ with $\text{supp}(u) = 3$ that are mutually orthogonal. Denote these circuits by*

$$\widetilde{\mathcal{C}}(B_c) := \{(1, -1, 0, 0, \dots, 0, 1), (0, 1, -1, 0, \dots, 0, 1), \dots, (0, 0, \dots, 0, 1, -1, 1)\}$$

where $s_i = (0, \dots, \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 1)$ for each $1 \leq i \leq c$. Consequently, if there exists a vector $v \in \mathcal{G}r(B_c)$ such that $v = (0, \dots, a, b, 0, \dots, \alpha)$ with $a > 0$, $b < 0$ and $\alpha \geq 1$, then v is reducible by one of the circuits and is therefore not a Graver basis element.

Denote by $\mathcal{G}r(\overline{B}_c) = \{(a_0, a_1, \dots, a_c, 0) \in \mathcal{G}r(B_c)\}$ the set of homogeneous Graver basis elements. These elements are exactly the homogeneous primitive partition identities of order $c+1$. In the following we will show that $\mathcal{G}r(B_c)$ is **connected** by moves from $\widetilde{\mathcal{C}}(B_c)$ in the following sense:

- We can move from each Graver basis element to any other Graver basis element by repeatedly adding or subtracting elements from $\widetilde{\mathcal{C}}(B_c)$

- each intermediate vector in this path is also a Graver basis element.

As convention, require that the last component $b \geq 0$ for every vector in $\mathcal{Gr}(B_c)$.

Lemma 7.11. *Any element $(a_0, a_1, \dots, a_c, b) \in \mathcal{Gr}(B_c)$ fulfills $|b| < c$.*

Proof. By Theorem 7.7, $\|(a_0, a_1, \dots, a_c, 0)\|_1 \leq 2(c-1)$. Suppose there exists an element $g = (a_0, a_1, \dots, a_c, b) \in \mathcal{Gr}(B_c)$ such that $b > c > 0$. We construct a nonzero element that reduces g . Since $b > c$, g must have components $g_i > 0$, $g_j < 0$, for $i < j$, i.e. $(\dots, g_i, \dots, g_j, \dots, b)$. Consider the vector

$$f = g - e_i + e_j - (j-i)e_{c+2} \in \ker(B_c).$$

By construction, $f \sqsubseteq g$ because $j-i \leq c$. Therefore, g is not a minimal element and the result follows. \square

Theorem 7.12. *The Graver basis of B_c is connected.*

Proof. Let $(a_0, a_1, \dots, a_c, b) \in \mathcal{Gr}(B_c)$. Then by Lemma 7.11 we have that $|b| \leq c$. Thus, suppose that $v = (\dots, 0, a, b, 0, \dots, \alpha) \in \mathcal{Gr}(B_c)$ with $a > 0$, $b \geq 0$ and $\alpha \geq 1$. We want to show that if $s = (\dots, 0, 1, -1, 0, \dots, 1)$ is a circuit in $\widetilde{\mathcal{C}}(B_c)$ then $v - s \in \mathcal{Gr}(B_c)$, where $a-1 \geq 0$, $b+1 > 0$. If this is true, we may iterate the procedure, subtracting (or adding) circuits from each element until the entire Graver basis $\mathcal{Gr}(B_c)$ is obtained.

If $v - s$ equals an existing element of the Graver basis, then $v - s$ is reducible and we are done. We will show that $x + s$ reduces v or $y + s$ reduces v contradicting

Figure 6: The Graver basis of B_4

the assumption that v is a Graver basis element. This then implies that $v - s$ is indecomposable, that is, a Graver basis element. Let $x = (\dots, 0, c, d, 0, \dots, \beta) \in \mathcal{Gr}(B_c)$ with $c > 0, d \geq 0$ and $y = (\dots, 0, e, f, 0, \dots, \gamma) \in \mathcal{Gr}(B_c)$ with $e > 0, f \geq 0$. Therefore, we have the conditions

$$0 \leq c, e \leq a - 1, \quad 0 \leq d, f \leq b + 1 \quad \text{and} \quad 0 \leq \beta, \gamma \leq \alpha - 1.$$

We have the following cases:

1. If $0 < d$, then $0 \leq c + 1 \leq a$ and $0 \leq d - 1 < b$ and $0 < \beta + 1 \leq \alpha$.
2. If $d = 0$ then $f = b + 1$ and $0 < e + 1 \leq a$, $0 < f - 1 \leq b$, $0 < \gamma + 1 \leq \alpha$.

All that needs to be shown is that the vector $y + s = (\dots, e + 1, f - 1, \dots, \gamma + 1)$ is not equal to the original vector $v \in \mathcal{Gr}(B_c)$. But, $c > 0$ so by the assumption that $x \sqsubseteq v - s$, $c + e = a - 1 \Rightarrow a - (e + 1) = c > 0$ and hence $(\dots, e + 1, f - 1, \dots, \gamma + 1) \neq v$. Therefore, $v - s \in \mathcal{Gr}(B_c)$ and this holds for any vector v which implies that $\mathcal{Gr}(B_c)$ is connected. \square

Connectivity gives an explicit structure to the Graver basis for B_c . See Figure 6 for a diagram of the Graver basis of B_4 . We may use this structure and the following lemma to give a bound on the degree of the elements in the Graver basis.

Lemma 7.13. *The 1-norm of elements $v \in \mathcal{Gr}(B_c)$ changes by ± 1 when any element $s_i \in \widetilde{\mathcal{C}}(B_c)$ is added to v .*

Proof. Let $v = (v_0, v_1, \dots, v_c, b) \in \mathcal{G}r(B_c)$. Choose the convention that $b \geq 0$ for every Graver basis element v . Thus the sign pattern for elements in $\mathcal{G}r(B_c)$ is $(\dots, \geq 0, \dots, \leq 0, \dots, > 0)$, where the "gaps" are given by zeros. For any $s_i \in \widetilde{\mathcal{C}}(B_c)$, consider

$$v + s_i = (\dots, v_i + 1, v_{i+1} - 1, \dots, b + 1),$$

where $b \neq 0$ and $v_i > 0$. Adding s_i effectively adds zero except for the +1 in the b component. Hence, the norm increases by ± 1 . \square

Remark 7.14. For $c > 2$, there is exactly one vector in the Graver basis of B_c that has the last component equal to c ; namely $w = (1, 0, \dots, -1, c)$. Moreover, for $s_i \in \widetilde{\mathcal{C}}(B_c)$, $w - s_1 = (0, 1, 0, \dots, -1, c - 1)$, $w - s_c = (1, 0, \dots, -1, 0, c - 1)$ and for $2 \leq k \leq c - 1$, $w - s_k$ are reducible.

Lemma 7.15. Let $v \in \mathcal{G}r(B_c)$ be given. The sign pattern of v determines whether or not the 1-norm increases or decreases when adding $s_i \in \widetilde{\mathcal{C}}(B_c)$.

Proof. Let $s_i \in \widetilde{\mathcal{C}}(B_c)$. By Remark 7.14, the sign patterns associated to an increase by one in the 1-norm are

$$(\dots, +, +, \dots) \quad \text{or} \quad (\dots, -, -, \dots) \quad (6)$$

$$(\dots, 0, +, \dots) \quad \text{or} \quad (\dots, -, 0, \dots) \quad (7)$$

The 1-norm decreases by one upon adding by s_i when v is of the form

$$(\dots, -, +, \dots) \quad \text{or} \quad (\dots, +, -, \dots) \quad (8)$$

□

Define by a "stage" the value of the last component b in a vector $v = (a_0, a_1, \dots, a_c, b) \in \mathcal{Gr}(B_c)$.

Lemma 7.16. *For $v \in \mathcal{Gr}(\overline{B_c})$ of maximal 1-norm $2c$, $v + a_i s_i$ for some $a_i \in \mathbb{Z}$ have 1-norm less than $2c$.*

Proof. By Theorem 7.7, the maximal 1-norm of elements in $\mathcal{Gr}(\overline{B_c})$ is $2c$, and the elements in $\widetilde{\mathcal{C}(B_c)}$ are circuits s_i with $\|s_i\|_1 = 3$. Consider sums and differences of the elements $s_i \in \widetilde{\mathcal{C}(B_c)}$ and $v \in \mathcal{Gr}(\overline{B_c})$. Any element $v + s_i$ with sign pattern $(\dots, +, -, \dots, +)$ will be reducible by some $s_j \in \widetilde{\mathcal{C}(B_c)}$ and therefore will not be a Graver basis element.

From Lemma 7.13, the 1-norm of elements $v + s_i$ will only change by ± 1 . Begin at stage $b = 0$ and add elements from $\widetilde{\mathcal{C}(B_c)}$ to generate all elements in stage $b = 1$. By Lemma 7.11, the number of stages is finite and equals c .

By Remark 7.14, the last two stages in the process are

$$\begin{array}{c} (1, 0, \dots, -1, c) \\ \nearrow \quad \nwarrow \\ (0, 1, 0, \dots, -1, c-1) \quad (1, 0, 0, \dots, -1, 0, c-1) \end{array}$$

where

$$\|(0, 1, 0, \dots, -1, c-1)\|_1 = \|(1, 0, 0, \dots, -1, 0, c-1)\|_1 = c+1 \quad (9)$$

$$< \|(1, 0, \dots, 0, -1, c)\|_1 = c+2 < 2c. \quad (10)$$

The elements $v = (v_0, v_1, \dots, v_c, 0) \in \mathcal{Gr}(\overline{H_c})$ of maximal 1-norm $2c$ are circuits where one entry is $|v_i| = c$. Since these elements have coordinate sum zero, they satisfy $v_1 + v_c = c$ with $v_1 \in \phi(c)$ where $\phi(n)$ is the Euler phi function. As a result of Lemma 7.13, the elements

$$\alpha_1 = (-(c-1), c, 0, \dots, 0, -1, 0) \quad \alpha_2 = (1, 0, \dots, -c, c-1, 0)$$

generate the following paths for $s_1 \in \widetilde{\mathcal{C}(B_c)}$: $\alpha_1 + cs_1 = (1, 0, \dots, -1, c)$ and $\alpha_2 + cs_{c-1} = (1, 0, \dots, -1, c)$. The addition of other elements $s_i \in \widetilde{\mathcal{C}(B_c)}$ are reducible and are therefore not Graver basis elements. By Lemma 7.15, both of these paths decrease the 1-norm by one exactly $c-1$ times and then increase by one in the last stage. Therefore, the vectors $\alpha_1 + a_i s_1$ and $\alpha_2 + a_i s_{c-1}$ for $1 \leq i \leq c$ have 1-norm less than $2c$. □

8 The Graver Complexity of a matrix $\mathcal{A} \subset \mathbb{Z}^{2 \times 4}$

Recall that the true circuits are the circuits for the Graver basis of \mathcal{A} both in the forms of $\mathcal{A} = (0, a, b, a + b)$ and in the general case.

Proposition 8.1. *Let $B = \mathcal{G}r(\mathcal{A})^T$ of the matrix $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & i_2 & i_3 & i_4 \end{pmatrix} \in \mathbb{Z}^{2 \times 4}$. Then any circuit $c \in \mathcal{C}(B)$ has $\text{supp}(c) = 3$.*

Proof. Columns of the matrix B lie in the kernel of \mathcal{A} , which is 2-dimensional. Thus, the $\text{rk}(B) = 2$ which implies that for any circuit $c \in \mathcal{C}(B)$, $\text{supp}(c) \leq \text{rk}(B) + 1 = 3$. □

Now it remains to show that the true circuits of the $\mathcal{G}r(\mathcal{A})$ have a linear bound.

Theorem 8.2. *Let $\mathcal{A} = (0, a, b, a + b)$. The maximal 1-norm of a true circuit in $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ is $2(a + b)$ and is taken on exactly for the vectors*

$$v = (0, (a + b) - j, 0, \dots, 0, a + b, 0, \dots, 0, j)$$

where the $a + b$ is the column of $(i, j)^T$ and the entries come from the deterministic definition of a true circuit.

Proof. By Proposition 4.7, the kernel of $\mathcal{G}r(\mathcal{A})^T$ is the same as of the $2 \times (c + 2)$

matrix given by the transpose of the Graver basis vectors

$$\begin{aligned} \mathcal{Gr}(\mathcal{A})^T &= \begin{pmatrix} \lambda & g_0 & g_0 + \lambda & g_0 + 2\lambda & \cdots & g_0 + c\lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 2 & \cdots & c \end{pmatrix} \end{aligned}$$

By definition of the support of a circuit $c \in \mathcal{C}(\mathcal{Gr}(\mathcal{A})^T)$, the $\text{supp}(c) \leq \text{rk}(M) + 1 = 3$.

Thus either the circuit with maximal 1-norm of a matrix is of the form $\begin{pmatrix} 1 & 1 & 0 \\ x & y & 1 \end{pmatrix}$

for any $x, y \in \mathbb{Z}$ is $\|(1, 1, y - x)\|$. In this case, $x = 0$ and $y = c$ which implies that

$\|(1, 1, y - x)\| = y - x + 2 \leq c - 0 + 2 \leq 2c$. Or there are matrices of the form $\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \end{pmatrix}$

where a circuit is of the form $(\pm(z - y), \pm(z - x), \pm(y - x))$ where $z = c$ and $x = 0$.

Thus the 1-norm is $\|(\pm(z - y), \pm(z - x), \pm(y - x))\| = 2z - 2x \leq 2c - 2 \cdot 0 = 2c$ with equality iff $z = c, x = 0$ and y can be anything. \square

The Graver representative is not necessarily unique. It can be found in the collection of circuits in $\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$ that are integer combinations of $\mathcal{C}(\mathcal{A})$.

Definition 8.3. Let $\mathcal{A} = (0, i_2, i_3, i_4)$, with $0 < i_2 < i_3 < i_4$. Denote by $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ the vectors in $\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$ that are integer combinations of circuits in $\mathcal{Gr}(\mathcal{A})$.

Then we have the following:

Proposition 8.4. Let $\mathcal{A} = (0, i_2, i_3, i_4)$, with $0 < i_2 < i_3 < i_4$. Then $\mathcal{C}(\mathcal{C}(\mathcal{A})) \subseteq \mathcal{C}(\mathcal{Gr}(\mathcal{Gr}(\mathcal{A})))$.

Proof. By Proposition 8.1, the support of any circuit in $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ is 3. Since $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{G}r(\mathcal{A})$, every element in $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ is an element in $\mathcal{C}(\mathcal{G}r(\mathcal{A})^T)$ if we place a zero in each component of a vector in $\mathcal{G}r(\mathcal{A})^T$ for all columns in $\mathcal{G}r(\mathcal{A})^T$ corresponding to vectors in $\mathcal{G}r(\mathcal{A}) \setminus \mathcal{C}(\mathcal{A})$. \square

Remark 8.5. *The reverse inclusion is not true. Unfortunately, not every element in $\mathcal{C}(\mathcal{G}r(\mathcal{A})^T)$ arises in this way. Consider the matrix $(0, 2, 5, 8)$. The Graver representatives for $(0, 2, 5, 8)$ are*

$$(5, -4, 0, 0, 0, 0, -1), (0, -3, 0, 5, 0, 0, -2), (0, -2, 0, 0, 5, 0, -3), (0, -1, 0, 0, 0, 5, -4)$$

where only $(5, -4, 0, 0, 0, 0, -1) \in \mathcal{C}(\mathcal{C}(\mathcal{A}))$.

We may now prove that the Graver complexity for a general matrix $\mathcal{A}' = (i_1, i_2, i_3, i_4)$ can be given in terms of the integers i_1, i_2, i_3, i_4 . Since \mathcal{A}' has kernel isomorphic to $\mathcal{A} = (0, i_2, i_3, i_4)$, we will use this form to prove the theorem. By Corollary 4.4, the integers of $\mathcal{A} = (0, a, b, a + b)$ were relatively prime and so were their respective differences. Thus, for general matrices $\mathcal{A} = (0, i_2, i_3, i_4)$, with $0 < i_2 < i_3 < i_4$ we expect the property of the integers defining \mathcal{A} having relatively prime differences to be needed when showing a tight bound.

Theorem 8.6. *Let \mathcal{A} be the homogeneous 2×4 matrix of the form*

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i_1 & i_2 & i_3 & i_4 \end{pmatrix}$$

such that $0 \leq i_1 < i_2 < i_3 < i_4$. Then the Graver complexity $g(\mathcal{A})$ is bounded above by the maximal 1-norm of the true circuits of \mathcal{A} by

$$\max\{i_2 + i_3 + i_4 - 3i_1, 3i_4 - i_1 - i_2 - i_3\}.$$

If, in addition, the set of integers $\{i_2 - i_1, i_3 - i_1, i_4 - i_1, i_3 - i_2, i_4 - i_2, i_4 - i_3\}$ are pairwise coprime, then the bound is tight.

Proof. These bounds equivalently hold if we write the matrix as $\mathcal{A} = (0, i_2, i_3, i_4)$ such that $0 < i_2 < i_3 < i_4$ because of elementary row operations.

Because the Graver basis $\mathcal{G}r(\mathcal{A})$ is bounded by circuits, the Graver representative in $\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$ will correspond to an element in $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ by Proposition 8.4. Assuming Hosten's True Circuit Conjecture (Conjecture 3.4), the elements in the Graver basis of the Graver basis are covered by the true circuits of the circuits of \mathcal{A} . Possible combinations of the numbers in $\mathcal{D}(\mathcal{A}) = \max\{i_1, i_2, i_3, i_4, i_2 - i_1, i_3 - i_1, i_4 - i_1, i_3 - i_2, i_4 - i_2, i_4 - i_3\}$ give the upper bound defined by the $\text{maxcircuit}(\mathcal{G}r(\mathcal{A}))$. Thus $\text{maxcircuit}(\mathcal{G}r(\mathcal{A})) = \max\{i_1 + i_2 + i_3 + i_4 = i_2 - i_1 + i_3 - i_1 + i_4 - i_1 = i_2 + i_3 + i_4 - 3i_1, i_4 - i_1 + i_4 - i_2 + i_4 - i_3 = 3i_4 - i_1 - i_2 - i_3\}$. Therefore, these give an upper bound for the Graver complexity of \mathcal{A} in terms of the true circuits for \mathcal{A} .

If all the differences are pairwise coprime, then $i_2 + i_3 + i_4 - 3i_1 = 3i_4 - i_1 - i_2 - i_3$ and the bound is tight. For example, consider the matrix $(0, 3, 7, 8)$ which has Graver representative $g = (0, 0, 8, 0, -7, 0, 0, 3)$ with Graver complexity $g(0, 3, 7, 8) = 18$. \square

Remark 8.7. *The matrix $(0, 1, 2, 3)$ that represents the twisted cubic curve in \mathbb{P}^3 has entries whose differences are relatively prime. The Graver representative is $(3, -2, 0, 0, 1)$ which implies that the Graver complexity is $g = 6$ so the bound is tight. However, it is not necessary for the differences of the entries to be coprime in order to obtain a tight bound. For example, consider $(0, 1, 7, 9)$ where the pairwise differences are not coprime. The Graver representative is $g = (9, -7, 0, 0, 0, 0, 1)$ and thus the Graver complexity is $g(0, 1, 7, 9) = 17$ which is a tight bound.*

There are other forms of the general matrix \mathcal{A} which have kernels that can be related to the matrix B_c for some c .

Lemma 8.8. *The matrix of the form*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & a & b & ab \end{pmatrix},$$

has Graver basis whose structure is a line given by $\mathcal{G}r(0, a, b, ab) = \{g_0, g_0 + \lambda, g_0 + 2\lambda, \dots, g_0 + a\lambda, \lambda\}$, where $\lambda = (1, -b, a, 0)$ and $g_0 = (-a, b, 0, -1)$

Proof. The two vectors in the lemma lie in the kernel of the Hermite normal form of the matrix $(0, a, b, ab)$ and thus generate a lattice. A similar proof to that of Lemma 4.7 shows that this is a set of minimal vectors and does, by the algorithm of either Hemmecke or Pottier, generate the Graver basis. □

Proposition 8.9. *The region bounded by the circuits corresponding to the Graver basis of the matrix $\mathcal{A} = (0, a, b, ab)$ forms a parallelogram whose area is $2a$.*

Proof. Construct the Gale dual $G_{\mathcal{L}}^*$ for the matrix \mathcal{A} . Denote by \mathcal{P} the region defined by connecting the lattice points corresponding to the circuits for $\mathcal{G}r(\mathcal{A})$. The Gale dual diagram is defined by two vectors of minimal length; we take the vectors $(1, -b, a, 0)$ and $(1 - a, 0, a, -1) = g_0 + \lambda$ to define $G_{\mathcal{L}}$. Then the Gale dual diagram will be given by the matrix

$$G_{\mathcal{L}}^* = \begin{pmatrix} 1 - a & -1 \\ 0 & b \\ a & -a \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 - a & -1 \\ 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}$$

after removing multiplicities. A simple transformation of one triangle creates a rectangle whose length is $(a - 1) + 1 = a$ and whose width is 2. Therefore, the area is $Area(\mathcal{P}) = 2a$. \square

From the structure of the Graver basis is the preceding lemma, the kernel of $(0, a, b, ab)$ is the same as that for the matrix B_a . Hence

Corollary 8.10. *Assuming the 2c Conjecture is true, the Graver complexity of $\mathcal{A} = (0, a, b, ab)$ will be $g(\mathcal{A}) = 2a$.*

Lemma 8.11. *Let \mathcal{A} be $(0, a, 2a - 1, a(2a - 1))$, where $a \geq 2$. Then $\mathcal{G}r(\mathcal{A})$ is covered by circuits. In particular, $\mathcal{G}r(\mathcal{A})$ is given by circuits.*

Proof. After reducing the matrix into its Hermite normal form, we get that the min-

imal generators for the kernel of \mathcal{A} over the integers is

$$\langle (a-1, -2a+1, a, 0), (a-1, 0, -a, 1) \rangle.$$

These vectors generate a lattice \mathcal{L} and, by critical pair completion, determine the Graver basis $\mathcal{G}r(\mathcal{A})$. Consider the gale dual diagram defined by these minimal generators

$$G_{\mathcal{L}}^* = \begin{pmatrix} 1-a & a-1 \\ 0 & -2a+1 \\ a & a \\ -1 & 0 \end{pmatrix}.$$

Notice first that it is sufficient to prove the claim for $a = 2$. Otherwise, for any $k > 0$ the elements in $G_{\mathcal{L}}^*$ are simply

$$G_{\mathcal{L}}^* = \begin{pmatrix} 1-(a+k) & (a+k)-1 \\ 0 & -2(a+k)+1 \\ a+k & a+k \\ -1 & 0 \end{pmatrix}$$

which just stretches the original vectors by k . Thus consider the case where $a = 2$:

$$G_{\mathcal{L}}^* = \begin{pmatrix} 1 & -3 & 2 & 0 \\ -1 & 0 & 2 & -1 \end{pmatrix}^T. \text{ The cones defined by the rows of } G_{\mathcal{L}}^* \text{ are unimodular}$$

and hence the only Hilbert basis elements are the extremal rays. Therefore the circuits form the Graver basis and $\mathcal{G}r(\mathcal{A})$ is covered by circuits. \square

The following proposition completely classifies the case where the Graver complexity $g(\mathcal{A}) = 4$ for a 2×4 integer matrix \mathcal{A} :

Proposition 8.12. *Let \mathcal{A} be $(0, a, 2a-1, a(2a-1))$, where $a \geq 2$. Then $\ker(\mathcal{G}r(\mathcal{A})) = \ker(B_2)$ and $g(\mathcal{A}) = 4$.*

Proof. By Lemma 8.11, the kernel of the matrix \mathcal{A} is

$\langle (a-1, -2a+1, a, 0), (a-1, 0, -a, 1) \rangle$. Thus the Graver basis for \mathcal{A} is

$$\begin{aligned} \mathcal{G}r(\mathcal{A}) &= \{(a-1, -2a+1, a, 0), (a-1, 0, -a, 1), (2a-2, -2a, 0, 1), (0, 2a-1, -2a, 1)\} \\ &= \{g_0, g_0 + \lambda, g_0 - \lambda, \lambda\} \end{aligned}$$

where $\lambda = (a-1, -2a+1, a, 0)$ and $g_0 = (a-1, 0, -a, 1)$. Rearranging these elements to form a new matrix with these as column vectors, we get the matrix

$$\begin{pmatrix} 2-2a & 1-a & 0 & a-1 \\ 2a & 0 & 1-2a & -2a+1 \\ 0 & a & 2a & a \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

whose kernel has minimal generating set $\{(0, 1, -1, 1), (-1, 1, 0, -1)\}$ over \mathbb{Z} . These are precisely the generators for the kernel of B_2 . By another application of the completion algorithm, we get that

$$\mathcal{G}r(B_2) = \mathcal{G}r(\mathcal{G}r(\mathcal{A})) = \{(0, 1, -1, 1), (-1, 1, 0, -1), (-1, 2, -1, 0), (-1, 0, 1, -2)\}.$$

The maximal 1-norm of any element in $\mathcal{G}r(B_2)$ is 4, hence $g(\mathcal{A}) = 4$. □

Now consider matrices of the form $(0, 1, 2, k)$.

Theorem 8.13. *For matrices of the form*

$$(0, 1, 2, k) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k \end{pmatrix}$$

the Graver complexity $g(0, 1, 2, k)$ is equal to $2(k - 1)$ when k is even and is $3(k - 1)$ when k is odd.

Proof. After row reduction to the matrix $(0, 1, 2, k)$, minimal generators for the kernel are $\{(k - 1, -k, 0, 1), (-1, 2, -1, 0)\}$. Let $\lambda = (1, -2, 1, 0)$, $g_0 = (k - 1, -k, 0, 1) \in \ker(0, 1, 2, k)$. A Graver basis for $\mathcal{G}r(0, 1, 2, k)$ is determined by two cases:

Case 1: If k is even then a Graver basis is

$$\mathcal{G}r(0, 1, 2, k) = \{g_0, g_0 + \lambda, g_0 + 2\lambda, \dots, g_0 + (k - 1)\lambda, \lambda\}.$$

Let $\{a_0, \dots, a_c, \beta\} \in \ker(\mathcal{G}r(0, 1, 2, k))$, where $c = k - 1$. Then by Theorem 4.8, the kernel of $\mathcal{G}r(0, 1, 2, k)$ is the same as that of B_{k-1} .

Case 2: If k is odd, then $(0, k - 2, k - 1, k)$ is equivalent to $(0, 1, 2, k)$. The g.c.d. of the pairwise differences of $\{k - 2, k - 1, k\}$ is one. Therefore, by Theorem 8.6, the Graver complexity will be tight and will equal $g(0, 1, 2, k) = k - 2 + k - 1 + k = 3(k - 1)$. \square

We state here only one of the cases for matrices of the form $(0, 1, 3, k)$:

Proposition 8.14. *For matrices of the form*

$$(0, 1, 3, k) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k \end{pmatrix}$$

the Graver complexity $g(0, 1, 3, k)$ is equal to $3k - 4$ when k is even and not divisible by 3.

Proof. The matrix $(0, 1, 3, k)$ is equivalent to the matrix $(0, k - 3, k - 1, k)$. Because the pairwise differences $\{k - 3, k - 1, k\}$ are all relatively prime, by Theorem 8.6 the bound is tight and equals the sum of the integers. Thus $g(0, 1, 3, k) = k - 3 + k - 1 + k = 3k - 4$. □

If k is odd and not divisible by 3, then the differences need not be pairwise coprime. For example, $(0, 1, 3, 7)$ is equivalent to $(0, 4, 6, 7)$ whose integers are not relatively prime.

9 Complete Bipartite Graphs

We recall some basic definitions from graph theory.

Definition 9.1. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_s\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_t\}$. The **incidence matrix** of G is the $s \times t$ matrix $I(G) = (a_{ij})$ where the a_{ij} is 1 if the edge e_j is incident with vertex v_i and 0 otherwise. The matrix $I(G)$ is **totally unimodular** if every square submatrix has determinant $\{0, \pm 1\}$. A graph G is **bipartite** if $V(G)$ can be partitioned into two nonempty subsets V_1, V_2 such that every edge of G joins a vertex of V_1 to every vertex of V_2 . A graph G is a **complete bipartite graph** if G is bipartite and each vertex of V_1 is adjacent to every vertex of V_2 .*

Consider a matrix $\mathcal{A} = \{1, 1, \dots, 1\}$ with n copies of $1 \in \mathbb{Z}$. The r -th Lawrence lifting $\mathcal{A}^{(r)}$ corresponds to two-dimensional tables of the format $n \times r$. The Graver basis of $\mathcal{A}^{(r)}$ is the set of all circuits of the complete bipartite graph $K_{n,r}$. It is a result of Poincaré that the edge-incidence matrix associated to a graph is unimodular. A nice result for unimodular matrices by Sturmfels [21]

Proposition 9.2. [21]

If \mathcal{A} is an unimodular matrix then $\mathcal{C}(\mathcal{A}) = \mathcal{G}r(\mathcal{A})$.

tells us, in particular, that the circuits are the Graver basis for complete bipartite graphs. Thus the structure of the Graver basis for $K_{m,n}$ is given by the circuits.

We compute the Graver complexity of certain complete bipartite graphs and begin

with some results for $K_{1,n}$, which is referred to as a **star**. Denote by $(m \times n)$ the incidence matrix associated to the complete bipartite graph $K_{m,n}$.

Proposition 9.3. *The Graver basis of the complete bipartite graph $K_{1,m}$ is isomorphic to that of K_m . Furthermore, the Graver complexity is $g(1 \times m) = m$.*

Proof. Let the matrix associated to K_m be \mathcal{A} . The matrix associated to $K_{1,m}$ is the Lawrence lifting $\Lambda(\mathcal{A})$ and its kernel is isomorphic to $\ker(\mathcal{A})$.

Finding the circuits of $K_{1,m}$ amounts to choosing one of the m vertices. For the paths to be closed walks, it is therefore necessary to have overlaps on the same path. Thus the circuits of $K_{1,m}$ are the same as those for K_m .

Since the $\ker(K_{1,m}) \cong \ker(K_m)$, the Graver complexity of the $(1 \times m)$ is the same as that of K_m . But K_m has matrix representative $\underbrace{(1, 1, 1, \dots, 1)}_m$ which has Graver complexity equal to m (see [18]). \square

Proposition 9.4. *The Graver complexity of $K_{2,m}$ is $g(K_{2,m}) = m$.*

Proof. Circuits in $\mathcal{G}r(K_{2,m}) = \mathcal{G}r(2 \times m)$ correspond to the entries in a 2×2 table and will have the form $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Thus every element in the Graver basis of $K_{2,m}$ will be of the form $\{u, -u\}$, where $u \in \mathbb{Z}^n$. Therefore, the kernel of the Graver matrix for $K_{2,m}$ is the kernel of the matrix of the directed graph K_m . This matrix has the rows are labeled $1, \dots, m$ and the columns are pairs (i, j) with the second one occurring in

each column is negative. Because the matrix is unimodular, the Graver basis elements are squarefree and therefore the degree of the basis elements is at most m .

To conclude the the degree of the Graver basis elements is exactly m , we must exhibit one of degree m . This representative is not unique. For $K_{2,8}$ the following is a Graver representative

$$(-1, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 1, 0, 0, 0, -1, 0, 0, 1)$$

in $\mathcal{Gr}\mathcal{Gr}(K_{2,8})$ which has maximal 1-norm equal to 8. Therefore the Graver degree of $K_{2,m}$ is m . \square

By considering linear different linear sections of the matrix $\mathcal{Gr}(\mathcal{A})^T$ for $K_{3,4}$ we found that

$(4, -4, 0, 0, 0, 0, -3, 0, 0, 3, 0, 0, 0, -6, 0, 0, 0, -2, 5)$ is a Graver representative and thus

Observation 9.5. *The Graver complexity of $K_{3,4}$ is $g(3 \times 4) \geq 27$.*

The complexity will change given different linear sections. We want to either prove or disprove this computational result.

For $K_{1,3}$, the Graver basis is $\mathcal{Gr}(1 \times 3) = \{v = (1, -1, 0), u = (1, 0, -1), u + v = (0, 1, -1)\}$ and the Graver complexity is $g(1 \times 3) = 3$ given by the only Graver representative $(1, -1, 1)$. These same elements $\{u, v, u + v\}$ are found in the Graver basis of both $K_{2,3}$, $K_{3,3}$:

$\mathcal{Gr}(2 \times 3)$	<i>Representative</i>
$(1, 0, -1, -1, 0, 1)$	$(u, -u)$
$(-1, 1, 0, 1, -1, 0)$	$(v, -v)$
$(0, 1, -1, 0, -1, 1)$	$(u + v, -u - v)$

In the case of $K_{3,3}$, there are 36 Graver representatives, in the 953 elements in $\mathcal{GrGr}(3 \times 3)$ of the form $(0, 0, 3, 0, 0, -2, -2, 1, 0, 0, 1, 0, 0, 0, 0)$, up to symmetry, and Graver complexity $g = 9$. The following chart describes the Graver basis of $K_{3,3}$ in terms of the Graver basis elements of $K_{1,3}$. The representative demonstrates the possible permutations under the symmetry group $S_3 \times S_3$.

$\mathcal{Gr}(3 \times 3)$	<i>Representative</i>
$(-1, 1, 0, 1, -1, 0, 0, 0, 0)$	$(-v, u + v, 0)$
$(0, 0, 0, 1, 0, -1, -1, 0, 1)$	$(0, u, -u)$
$(-1, 0, 1, 1, 0, -1, 0, 0, 0)$	$(-u, u, 0)$
$(1, -1, 0, 0, 0, 0, -1, 1, 0)$	$(v, 0, -v)$
$(0, 0, 0, 1, -1, 0, -1, 1, 0)$	$(0, v, -v)$
$(-1, 1, 0, 1, 0, -1, 0, -1, 1)$	$(-v, u, u + v)$
$(0, -1, 1, 1, 0, -1, -1, 1, 0)$	$(-u - v, u, -v)$
$(1, -1, 0, 0, 1, -1, -1, 0, 1)$	$(v, u + v, -u)$
$(-1, 0, 1, 0, 0, 0, 1, 0, -1)$	$(-u, 0, u)$
$(0, -1, 1, 0, 1, -1, 0, 0, 0)$	$(-u - v, u + v, 0)$
$(0, 1, -1, 1, -1, 0, -1, 0, 1)$	$(u + v, v, -u)$
$(0, 0, 0, 0, 1, -1, 0, -1, 1)$	$(0, u + v, -u - v)$
$(0, -1, 1, 0, 0, 0, 0, 1, -1)$	$(-u - v, 0, u + v)$
$(-1, 0, 1, 0, 1, -1, 1, -1, 0)$	$(-u, u + v, v)$
$(-1, 0, 1, 1, -1, 0, 0, 1, -1)$	$(-u, v, u + v)$

A natural question to ask is whether it is possible to predict the nonzero coordinates just given the Graver basis. For $K_{3,3}$ we make the following remark by using one of the many Graver representatives:

Remark 9.6. *The binomials $x_1x_6x_8 - x_2x_4x_9$ and $x_1x_6x_8 - x_3x_5x_7$ corresponding to vectors in $\mathcal{Gr}(3 \times 3)$ represent the same graph up to symmetry. The same is true for the binomials $x_1x_5x_9 - x_2x_6x_7$ and $x_1x_5x_9 - x_3x_4x_8$. The other two cycles of length*

6 do not have symmetric representatives. There are Graver representatives that have nonzero components corresponding to one but not both of the vectors defining the same graph.

The elements in the Graver basis of $K_{2,3}$, $K_{3,3}$ are merely all the possible combinations of the elements in $\mathcal{Gr}(1 \times 3)$ up to symmetry. This pattern does not extend for the cases $K_{1,4}$, $K_{2,4}$, $K_{3,4}$ simply because the dimensions of the matrices are not conducive to the setup. For example, consider the following tables where the representative for the Graver basis of $K_{2,4}$ is written in terms of $K_{1,4}$.

$\mathcal{Gr}(1 \times 4)$	Representative	$\mathcal{Gr}(2 \times 4)$	Representative
$(-1, 1, 0, 0)$	a	$(1, 0, 0, -1, -1, 0, 0, 1)$	$(c, -c)$
$(1, 0, -1, 0)$	$-b$	$(-1, 0, 1, 0, 1, 0, -1, 0)$	$(b, -b)$
$(0, 1, -1, 0)$	$a - b$	$(-1, 1, 0, 0, 1, -1, 0, 0)$	$(a, -a)$
$(1, 0, 0, -1)$	c	$(0, 0, 1, -1, 0, 0, -1, 1)$	$(b + c, -b - c)$
$(0, 1, 0, -1)$	$a + c$	$(0, 1, 0, -1, 0, -1, 0, 1)$	$(a + c, -a - c)$
$(0, 0, 1, -1)$	$c + b$	$(0, 1, -1, 0, 0, -1, 1, 0)$	$(a - b, b - a)$

Lemma 9.7. *Let $K_{3,4}$ be a complete bipartite graph. The Graver basis $\mathcal{Gr}(3 \times 4)$ consists of circuits of length 4 or 6.*

Proof. The elements in the Graver basis $\mathcal{Gr}(3 \times 4)$ lie in \mathbb{Z}^{12} and determine 3×4 tables. Any circuit will consist of either 2 vertices from m and 2 from n or 3 vertices from both m and n . Every element in the Graver basis for the corresponding 7×12 matrix can be written in the form (a, b, c) where $a, b, c \in \{0, \pm u, \pm v, \pm w, \pm u + v, \pm u + w, \pm v + w\}$

and each of these elements u, v, w have support equal to 2. Thus the maximal 1-norm of a circuit in $\mathcal{G}r(3 \times 4)$ is 6. \square

Recall that the circuits of a unimodular matrix are the same as the Graver basis elements.

Corollary 9.8. *For the complete bipartite graph $K_{m,n}$, the circuits have maximal 1-norm equal to $2m$.*

10 Tables

10.1 General Matrices $\mathcal{A} = (0, i_2, i_3, i_4)$

$(0, i_1, i_2, i_3)$	$\#\mathcal{Gr}(\mathcal{A})$	$\#\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$	$g(\mathcal{A})$	<i>Representative</i>
(0, 2, 3, 6)	4	4	4	(1, -2, 0, -1)
(0, 1, 2, 3)	5	13	6	(3, -2, 0, 0, 1)
(0, 1, 2, 4)	5	13	6	(0, 2, -3, 0, 1)
(0, 2, 3, 4)	5	13	6	(3, -2, 0, 0, -1)
(0, 1, 3, 4)	6	36	8	(4, -3, 0, 0, 0, 1)
(0, 1, 3, 9)	6	36	8	(0, 3, -4, 0, 0, 1)
(0, 6, 8, 9)	6	36	8	(3, 0, -4, 0, 0, 1)
(0, 1, 3, 5)	6	42	9	(5, -3, 0, 0, 0, 1)
(0, 1, 3, 6)	6	42	9	(0, 3, -5, 0, 0, 1)
(0, 1, 4, 6)	6	42	9	(0, 1, 0, 0, -5, 3)
(0, 2, 4, 5)	6	42	9	(0, 3, -5, 0, 0, 1)
(0, 2, 5, 6)	6	42	9	(5, -3, 0, 0, 0, -1)
(0, 3, 4, 8)	6	42	9	(0, 5, 0, 3, 0, -1)
(0, 3, 5, 6)	6	42	9	(0, 3, -5, 0, 0, -1)
(0, 3, 5, 9)	6	42	9	(5, -3, 0, 0, 0, -1)
(0, 4, 5, 8)	6	42	9	(0, 3, 5, 0, 0, -1)
(0, 4, 6, 9)	6	42	9	(0, 5, 0, -3, 0, 1)
(0, 1, 2, 5)	7	95	12	(0, 3, 0, 0, 0, 5, -4)
(0, 2, 3, 8)	7	95	12	(5, 0, 0, 0, 4, 0, -3)
(0, 3, 4, 5)	7	95	12	(5, 0, 0, 0, 0, -4, 3)

$(0, i_1, i_2, i_3)$	$\#\mathcal{Gr}(\mathcal{A})$	$\#\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$	$g(\mathcal{A})$	<i>Representative</i>
$(0, 5, 6, 8)$	7	95	12	$(4, 0, 0, 0, 5, 0, -3)$
$(0, 1, 2, 6)$	7	101	10	$(0, 4, -5, 0, 0, 0, 1)$
$(0, 1, 4, 5)$	7	101	10	$(5, -4, 0, 0, 0, 0, 1)$
$(0, 3, 4, 9)$	7	101	10	$(5, 0, 0, 0, 3, 0, -2)$
$(0, 2, 3, 5)$	7	101	10	$(5, 0, 0, 0, 0, -3, 2)$
$(0, 2, 5, 8)$	7	101	10	$(5, -4, 0, 0, 0, 0, -1)$
$(0, 3, 6, 8)$	7	101	10	$(0, 4, -5, 0, 0, 0, 1)$
$(0, 4, 5, 6)$	7	101	10	$(4, 0, -5, 0, 0, 0, 1)$
$(0, 5, 6, 9)$	7	101	10	$(4, 5, 0, 0, 0, 0, -1)$
$(0, 1, 3, 7)$	7	127	13	$(0, 1, 0, 0, 0, 7, -5)$
$(0, 1, 4, 7)$	7	127	13	$(0, 1, 0, 0, 0, -7, 5)$
$(0, 1, 4, 8)$	7	127	13	$(0, 1, 0, 0, 0, -7, 5)$
$(0, 1, 5, 7)$	7	127	13	$(7, -5, 0, 0, 0, 0, 1)$
$(0, 1, 6, 8)$	7	127	13	$(7, -5, 0, 0, 0, 0, 1)$
$(0, 2, 3, 9)$	7	127	13	$(0, 5, -7, 0, 0, 0, 1)$
$(0, 2, 6, 7)$	7	127	13	$(0, 1, 0, 0, 0, 7, -5)$
$(0, 2, 6, 9)$	7	127	13	$(0, 5, -7, 0, 0, 0, 1)$
$(0, 2, 7, 8)$	7	127	13	$(0, 1, 0, 0, 0, 7, -5)$
$(0, 3, 6, 7)$	7	127	13	$(0, 5, -7, 0, 0, 0, 1)$
$(0, 3, 7, 9)$	7	127	13	$(0, 1, 0, 0, 0, -7, 5)$

$(0, i_1, i_2, i_3)$	$\#\mathcal{Gr}(\mathcal{A})$	$\#\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$	$g(\mathcal{A})$	<i>Representative</i>
$(0, 4, 6, 7)$	7	127	13	$(0, 0, 0, 7, 0, -5, 1)$
$(0, 4, 7, 8)$	7	127	13	$(0, 5, -7, 0, 0, 0, -1)$
$(0, 6, 7, 9)$	7	127	13	$(5, 0, -7, 0, 0, 0, 1)$
$(0, 1, 6, 9)$	7	142	14	$(0, 1, 0, 0, 0, -3, -5)$
$(0, 3, 8, 9)$	7	142	14	$(0, 5, -8, 0, 0, 0, -1)$
$(0, 1, 5, 6)$	8	243	12	$(6, -5, 0, 0, 0, 0, 1)$
$(0, 2, 3, 7)$	8	269	16	$(0, 7, 0, 0, 0, 0, 4, -5)$
$(0, 4, 5, 7)$	8	269	16	$(7, 0, 0, 0, 0, 0, -5, 4)$
$(0, 2, 4, 7)$	8	273	15	$(0, 3, 0, 0, 0, 0, 7, -5)$
$(0, 3, 5, 7)$	8	273	15	$(0, 7, 0, -5, 0, 0, 0, 3)$
$(0, 1, 5, 8)$	8	354	18	$(0, 3, 0, 0, 0, 0, 8, -7)$
$(0, 1, 5, 9)$	8	354	17	$(0, 1, 0, 0, 0, 0, -9, 7)$
$(0, 1, 7, 9)$	8	354	17	$(9, -7, 0, 0, 0, 0, 0, 1)$
$(0, 2, 8, 9)$	8	354	17	$(0, 1, 0, 0, 0, 0, 9, -7)$
$(0, 3, 7, 8)$	8	354	18	$(0, 0, 8, 0, -7, 0, 0, 3)$
$(0, 4, 8, 9)$	8	354	17	$(0, 7, -9, 0, 0, 0, 0, 1)$
$(0, 1, 2, 7)$	9	604	18	$(0, 5, 0, 0, 0, 0, 0, 7, -6)$
$(0, 5, 6, 7)$	9	604	18	$(7, 0, 0, 0, 0, 0, 0, -6, 5)$
$(0, 1, 2, 8)$	9	611	14	$(0, 6, -7, 0, 0, 0, 0, 0, 1)$
$(0, 1, 6, 7)$	9	611	14	$(7, -6, 0, 0, 0, 0, 0, 0, 1)$

$(0, i_1, i_2, i_3)$	$\#\mathcal{Gr}(\mathcal{A})$	$\#\mathcal{Gr}(\mathcal{Gr}(\mathcal{A}))$	$g(\mathcal{A})$	<i>Representative</i>
$(0, 2, 5, 7)$	9	611	14	$(7, 0, 0, -5, 0, 0, 0, 0, 2)$
$(0, 3, 4, 7)$	9	611	14	$(7, 0, 0, 0, 0, 0, 0, -4, 3)$
$(0, 6, 7, 8)$	9	611	14	$(6, 0, -7, 0, 0, 0, 0, 0, 1)$
$(0, 1, 3, 8)$	9	647	20	$(0, 5, 0, 0, 0, 0, 0, 0, 8, -7)$
$(0, 5, 7, 8)$	9	647	20	$(0, 8, 0, 0, 0, 0, 0, -7, 5)$
$(0, 2, 5, 9)$	9	730	20	$(0, 0, 0, 4, 0, 0, 0, 9, -7)$
$(0, 4, 7, 9)$	9	730	20	$(0, 9, 0, -7, 0, 0, 0, 0, 4)$
$(0, 1, 4, 9)$	9	906	22	$(0, 5, 0, 0, 0, 0, 0, 0, 9, -8)$
$(0, 5, 8, 9)$	9	906	22	$(0, 5, 0, 0, 0, 0, 0, 0, 9, -8)$
$(0, 3, 5, 8)$	10	1394	16	$(0, 8, 0, 0, 0, 0, 0, 0, -5, 3)$
$(0, 1, 7, 8)$	10	1394	16	$(8, -7, 0, 0, 0, 0, 0, 0, 0, 1)$
$(0, 2, 4, 9)$	10	1438	21	$(0, 5, 0, 0, 0, 0, 0, 0, 9, -7)$
$(0, 5, 7, 9)$	10	1438	21	$(0, 9, 0, 0, 0, 0, 0, 0, -7, 5)$
$(0, 1, 8, 9)$	11	3120	18	$(9, -8, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
$(0, 2, 7, 9)$	11	3120	18	$(9, 0, 0, -7, 0, 0, 0, 0, 0, 2)$
$(0, 4, 5, 9)$	11	3120	18	$(9, 0, 0, 0, 0, 0, 0, 0, 0, -5, 4)$
$(0, 1, 2, 9)$	11	3262	24	$(0, 7, 0, 0, 0, 0, 0, 0, 0, 9, -8)$
$(0, 7, 8, 9)$	11	3262	24	$(9, 0, 0, 0, 0, 0, 0, 0, 0, -8, 7)$

10.3 Matrices $\mathcal{A} = (0, a, b, ab)$

$(0, a, b, ab)$	$\#Gr(\mathcal{A})$	$\#Gr(Gr(\mathcal{A}))$	$g(\mathcal{A})$	<i>Graver Representative</i>
$(0, 2, 3, 6)$	4	4	4	$(1, -2, 0, -1)$
$(0, 3, 5, 15)$	4	4	4	$(-1, 2, 0, -1)$
$(0, 4, 7, 28)$	4	4	4	$(-1, 2, 0, -1)$
$(0, 5, 9, 45)$	4	4	4	$(-1, 2, 0, -1)$
$(0, 6, 11, 66)$	4	4	4	$(-1, 2, 0, -1)$
$(0, 2, 4, 8)$	5	13	6	$(0, -1, 3, 0, 0, 1)$
$(0, 3, 4, 12)$	5	13	6	$(2, 0, -3, 0, -1)$
$(0, 3, 7, 21)$	5	13	6	$(2, -3, 0, 0, 1)$
$(0, 5, 7, 35)$	5	13	6	$(2, 0, -3, 0, -1)$
$(0, 2, 5, 10)$	6	36	8	$(-3, 4, 0, 0, 0, -1)$
$(0, 4, 5, 20)$	6	36	8	$(-3, 0, 4, 0, 0, 1)$
$(0, 2, 6, 12)$	6	42	9	$(0, 3, -5, 0, 0, -1)$
$(0, 3, 6, 18)$	7	101	10	$(0, -4, 5, 0, 0, 1)$
$(0, 4, 6, 24)$	7	101	10	$(4, 0, -5, 0, 0, 0, -1)$
$(0, 5, 6, 30)$	7	101	10	$(4, 0, -5, 0, 0, 0, -1)$
$(0, 5, 8, 40)$	7	127	13	$(1, 0, 0, 0, 0, -7, 5)$
$(0, 2, 7, 14)$	8	243	12	$(-5, 6, 0, 0, 0, 0, 0, -1)$
$(0, 6, 7, 42)$	8	243	12	$(-5, 0, 6, 0, 0, 0, 0, 1)$
$(0, 6, 10, 60)$	8	354	17	$(7, 0, -9, 0, 0, 0, 0, -1)$
$(0, 4, 8, 32)$	9	611	14	$(0, -6, 7, 0, 0, 0, 0, 0, 1)$
$(0, 6, 8, 48)$	9	611	14	$(6, 0, -7, 0, 0, 0, 0, 0, -1)$

10.4 Other Special forms of the matrix \mathcal{A}

$(0, a, b, 2(a + b))$	$\#\mathcal{G}r(\mathcal{A})$	$\#\mathcal{G}r(\mathcal{G}r(\mathcal{A}))$	$g(\mathcal{A})$	<i>Representative</i>
$(0, 1, 4, 10)$	6	42	9	$(0, 3, -5, 0, 0, 1)$
$(0, 1, 2, 6)$	7	101	10	$(0, 4, -5, 0, 0, 0, 1)$
$(0, 2, 5, 14)$	7	127	13	$(5, -7, 0, 0, 0, -1)$
$(0, 2, 3, 10)$	8	269	16	$(7, 0, 0, 0, 0, 0, 5, -4)$
$(0, 1, 3, 8)$	9	647	20	$(0, 5, 0, 0, 0, 0, 0, 8, -7)$
$(0, 3, 4, 14)$	9	923	23	$(0, 7, 0, 0, 0, 0, -11, 0, 5)$
$(0, 4, 7, 22)$	9	946	21	$(9, -11, 0, 0, 0, 0, 0, 0, -1)$
$(0, 1, 6, 14)$	9	1100	24	$(0, 4, 0, 0, 0, 0, -13, 0, 7)$
$(0, 1, 7, 16)$	9	1461	26	$(0, 9, -16, 0, 0, 0, 0, 0, 1)$
$(0, 2, 7, 18)$	10	2102	28	$(11, 0, 0, 0, 0, 0, 0, 0, 9, -8)$
$(0, 4, 5, 18)$	10	2249	29	$(0, 9, 0, 0, 0, 0, 0, 0, 13, 0, -7)$
$(0, 1, 5, 12)$	10	2284	30	$(0, 7, 0, 0, 0, 0, 0, 0, 0, 12, -11)$
$(0, 5, 8, 26)$	10	2386	25	$(0, 0, 0, 0, 0, 0, 1, 0, 0, 13, -11)$
$(0, 1, 8, 18)$	10	3118	31	$(0, 5, 0, 0, 0, 0, 0, 0, -17, 0, 9)$
$(0, 2, 9, 22)$	11	4713	34	$(13, 0, 0, 0, 0, 0, 0, 0, 0, 0, 11, -10)$
$(0, 5, 6, 22)$	11	6049	36	$(0, 11, 0, 0, 0, 0, 0, 0, 0, -17, 0, 8)$
$(0, 3, 5, 16)$	11	6059	40	$(0, 0, 11, 0, 0, 0, 0, 0, 0, 0, 16, -13)$
$(0, 3, 8, 22)$	11	6899	37	$(0, 0, 7, 0, 0, 0, 0, 0, 0, 0, -19, 11)$
$(0, 1, 9, 20)$	12	19041	50	$(0, 11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 20, -19)$
$(0, 3, 7, 20)$	12	15432	50	$(0, 0, 0, 0, 13, 0, 0, 0, 0, 0, 0, 20, -17)$
$(0, 4, 9, 26)$	12	12833	41	$(17, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 13, -11)$
$(0, 6, 7, 26)$	12	140000	42	$(0, 13, 0, 0, 0, 0, 0, 0, 0, 0, 19, 0, -10)$
$(0, 7, 8, 30)$	13	32513	49	$(0, 15, 0, 0, 0, 0, 0, 0, 0, 0, 0, -23, 0, 11)$
$(0, 5, 7, 24)$	13	38150	60	$(0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0, 0, 17, -19)$

10.5 Integer Programming Relationship

Definition 10.1. *Let*

$$B_c = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & \cdots & c & 1 \end{pmatrix} = \left(H_c \left| \begin{array}{c} 1 & 0 \\ 0 & 1 \end{array} \right. \right).$$

Let π_c be the **projection** onto the first c components.

$$\pi_c(B_c) = \pi_c(H_c) + \pi_c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that the last two columns are the generators for the lattice \mathbb{Z}^2 .

We can use the relationship between the two matrices B_c and H_c to restate some integer programming problems:

$\mathcal{G}r(H_c)$ is a universal test set for

$$\min\{q^T z : H_c \cdot z = b, z \in \mathbb{Z}_+^c\}.$$

$\mathcal{G}r(B_c)$ is a universal test set for

$$\min\{q^T z : H_c \cdot z + u = b, z \in \mathbb{Z}_+^c, u \in \mathbb{Z}_+^2\}.$$

$\pi_c(\mathcal{G}r(B_c))$ is a universal test set for

$$\min\{q^T z : H_c \cdot z \leq b, z \in \mathbb{Z}_+^c\}$$

and also for

$$\min\{q^T z : H_c \cdot z \geq b, z \in \mathbb{Z}_+^c\}.$$

Therefore, there are the correspondences

$$H_c \leftrightarrow H_c \cdot z = 0 \text{ (or } H_c = b), \text{ where } z \geq 0 \quad (11)$$

$$B_c \leftrightarrow B_c \cdot z \geq 0 \text{ (or } H_c \geq b), \text{ where } z \geq 0 \quad (12)$$

References

- [1] A.I. Barvinok and K. Woods. Short rational generating functions for lattice point problems. *e-print*, 2002. <http://arxiv.org/abs/math.CO/0211146>.
- [2] Dave Bayer, Sorin Popescu, and Bernd Sturmfels. Syzygies of unimodular Lawrence ideals. *J. Reine Angew. Math.*, 534:169–186, 2001.
- [3] Michel Brion. Points entiers dans les polyèdres convexes. *Ann. Sci. École Norm. Sup. (4)*, 21(4):653–663, 1988.
- [4] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.
- [5] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.
- [6] J. De Loera, R. Hemmecke, J. Tauzer, and R. Yoshida. Latte. 2002. <http://www.math.ucdavis.edu/latte>.
- [7] Robert T. Firla. The design of exponential neighborhoods - a primal approach to integer programming. *Thesis, e-print*, 2002. <http://www.math.uni-magdeburg.de/firla/publications.html>.
- [8] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

- [9] Jack E. Graver. On the foundations of linear and integer linear programming. I. *Math. Programming*, 9(2):207–226, 1975.
- [10] Raymond Hemmecke. On the positive sum property of graver test sets. *Preprint*, 2000.
- [11] Raymond Hemmecke. 4ti2. *Software package for computing Graver and Hilbert bases*, 2002. <http://www.4ti2.de>.
- [12] Serkan Hosten. Degrees of groebner bases of integer programs. *Thesis, e-print*, 1997. <http://fener.sfsu.edu/~serkan>.
- [13] H. W. Lenstra, Jr. Integer programming with a fixed number of variables. *Math. Oper. Res.*, 8(4):538–548, 1983.
- [14] S. L'vovsky. On inflection points, monomial curves, and hypersurfaces containing projective curves. *Math. Ann.*, 306(4):719–735, 1996.
- [15] Mircea Mustața. Vanishing theorems on toric varieties. *Tohoku Math. J. (2)*, 54(3):451–470, 2002.
- [16] Tadao Oda. *Convex bodies and algebraic geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.
- [17] Loïc Pottier. The euclidean algorithm in dimension n . *Preprint*, 1996.
- [18] Francisco Santos and Bernd Sturmfels. Higher lawrence configurations. *e-print*, 2002. <http://arxiv.org/abs/math.CO/0209326>.

- [19] Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons Ltd., Chichester, 1986. A Wiley-Interscience Publication.
- [20] M. Stillman and D. Grayson. *Macaulay2. Software package*.
<http://www.math.uiuc.edu/Macaulay2>.
- [21] Bernd Sturmfels. *Gröbner bases and convex polytopes*. American Mathematical Society, Providence, RI, 1996.
- [22] Günter M. Ziegler. *Lectures on polytopes*. Springer-Verlag, New York, 1995.